BULLETIN OF THE UNIVERSITY OF WISCONSIN

No. 350: High School Series, No. 8

THE HIGH SCHOOL COURSE IN MATHEMATICS

BY

ERNEST B. SKINNER

Assistant Professor of Mathematics

The University of Wisconsin

The University of Wisconsin

MADISON

1910

HIGH SCHOOL SERIES

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PREFACE

The purpose of this bulletin is twofold: First, to outline the course in high school mathematics as well as to indicate methods of presentation for the various topics; second, to point out some of the subjects which lie just beyond the scope of the high school work and which form a necessary background for it. It is assumed that the best presentation of such topics as the fundamental laws of algebra, irrational numbers, the axioms and postulates of geometry, and limits, is impossible without a much fuller knowledge of these subjects on the part of the teacher than can be gotten from the ordinary elementary text-book.

To carry out the second purpose it has been necessary to introduce a considerable amount of material which can not be utilized in the class-room. It is hoped, however, that by means of the free use of foot-notes, the material has been so arranged that no teacher will be in doubt as to what to present and what not to present to his pupils.

Correspondence and criticism from the teachers of the state are cordially invited to the end that the mutually helpful relations now existing between the high schools and the department of mathematics at the university may be strengthened as far as possible.

To Mr. H. L. Terry, state high school inspector, to Professor E. C. Elliott, director of the course for the training of teachers, and to my colleagues in the department of mathematics, I am indebted for many helpful criticisms and suggestions.

Ernest B. Skinner.

The University of Wisconsin, Madison, July 1, 1909.

I

OBJECTS TO BE ATTAINED THROUGH THE STUDY OF MATHEMATICS

In discussing the objects to be attained by the study of mathematics one naturally puts the acquisition of useful knowledge in the foreground. A reasonable acquaintance with the rules of arithmetic and its application to the affairs of daily life, a knowledge of the more important mensuration formulas with the ability to apply the proper formula on short notice and with reasonable accuracy, ought to be a part of the education of every citizen. This ability needs to go a good way beyond mere skill in doing the sums necessary to cast up a butcher's or a grocer's bill.

So far as the pupil has to do with the facts of mathematics the important thing is to interpret and apply formulas that are given to him. To know that a formula is a shorthand expression for a statement, that translated into words becomes a rule by means of which important numerical results are found, to be able to furnish the translation of the formula and to apply the rule in a concrete case so as to express the desired result approximately or exactly in numerical units, is an accomplishment of no inconsiderable value.

Closely connected with this work is the matter of skill in literal arithmetic, or "*Buchstaben-Rechnung*" as it is called in the German schools—one of the important ends to be secured in high school mathematics. To use a familiar example: Every problem that can possibly arise in simple interest and simple discount can be solved by means of the two formulas,

$$I=Prt,$$
$$A=P(1+rt).$$

The power to interpret such formulas and to work with the symbols they contain is well-nigh indispensable. The ability to work with symbols demands that the pupil should have some skill in the manipulation of mathematical expressions.

But the teacher who proceeds upon the theory that the information to be derived from the study of mathematics, and

the ability to apply a few simple formulas to the determination of areas and volumes constitute the sole or even the chief benefits to be derived from such study makes a very serious mistake. For nine-tenths of the pupils who go through the high school a mastery of the four fundamental operations of arithmetic, a knowledge of the simpler problems of denominate numbers and of the simpler cases of interest, are sufficient for the needs of all the ordinary affairs of life. There are probably less than half a hundred independent formulas in the most advanced text-book in high school physics, and a boy of ordinary intelligence should be able to handle all the shop formulas used in any large manufacturing establishment after a year's work in algebra. Few people ever need more than half a dozen in mensuration formulas, and it is not at all necessary that one should know how to demonstrate these in order to use them. The mechanic has his shop card which answers practically all his mathematical questions; the bank clerk has his interest tables which furnish desired results at a glance; even the trained engineer has his pocket manual from which he is able to take many results for problems that arise under ordinary conditions without any computation whatever.

It may be admitted frankly that if the study of mathematics has no other object than to equip the pupil with knowledge that shall have some direct application in earning a livelihood, the amount of time given to the subject can scarcely be justified. There are indeed many other benefits that can be enumerated.

The ability to reason correctly from given hypotheses is one of the most important accomplishments that any person can acquire. The use of mathematics as an agency for cultivating this power has been questioned many times by high authorities but the fact that after nearly a century, the position of the subject in the elementary and secondary schools is stronger than ever, is tolerably good evidence that thinking people do not agree with the few who decry mathematics. Such uses as have been indicated in the preceeding paragraphs, valuable though they are, would hardly suffice to give such an important place in the curriculum to a subject which is distasteful to so many people.

The fundamental concepts of mathematics are relatively

few in number and are as a rule simple, and they lie tolerably close to the daily experience of the average person. The processes are direct and easy to understand and the form is well nigh perfect. For these reasons the subject would seem to be peculiarly fitted to train the mind in drawing correct conclusions from simple premises. The high school pupil should have made considerable progress in this direction by the time he is ready for his diploma.

The ability not only to appreciate the nature of mathematical proof, but to carry forward a demonstration while standing on his feet and under a fire of questions from the teacher and the class is a most valuable acquisition for the high school pupil. No subject taught in the high school is so well adapted to secure this end as mathematics and the careful teacher will see to it that everything possible is done to train pupils in this direction.

The ability to state a given problem, whether it be mathematical or not, in its simplest form is an invaluable accomplishment to any one. How often we fail to comprehend what another has in mind simply because the speaker lacks the ability to present the salient features of a given situation in order, giving to each point its due weight. It is just this separation of a problem into its simplest elements and placing these in order that mathematics seeks to accomplish, and the pupil who has completed his high school course should have made considerable progress in this direction.

It goes without saying that accuracy of expression is one of the things the teacher of mathematics should seek to inculcate. Mathematical truth is based upon definitions and axioms accurately formulated, and is capable of being expressed with a degree of accuracy that cannot be approached by any science not essentially mathematical in its form. It would be absurd to insist that everything that is spoken or written should be put into the form of a proposition in geometry, but the teacher of mathematics can do very much to correct the prevailing slovenliness and inaccuracy of expression and to cultivate a wholesome respect for truth expressed in a truthful fashion.

Professor Felix Klein has recently laid great stress upon the use of what may be called the "function idea" in elemen-

tary mathematics. The definition of a function presupposes the existence of a magnitude which varies in some definite way, as time which we suppose to increase uniformly, or the distance of a moving point from a fixed point. A function may then be defined as a variable which depends for its value upon another given variable. This definition makes the function a definite mathematical expression in which the variable appears. For example: The interest of a given principal at a given rate depends upon the time during which the principal remains at interest. The interest or *prt*, is then a function of the time in the sense that it depends upon the time, i. e., it changes as the time changes. Again, the space through which a body falls from rest under the influence of gravity alone is given by

$$S=\tfrac{1}{2}gt^2.$$

Here S, or $\frac{1}{2}gt^2$ is again a function of the time t, but the function $\frac{1}{2}gt^2$ differs radically from the function *prt* which occurs in the interest formula.

The volume of a gas at constant temperature is a function of the pressure to which it is subjected and again the function, $\frac{ct}{p}$ is different. If the temperature is supposed to vary also we have a function of two variables, p and t. There are a large number of such functions that occur in commercial arithmetic, in physics, in chemistry and in many other domains of thought, and a surprisingly large number of these are special cases of one of the two forms

$$\frac{ax+b}{cx+d} \text{ and } ax^2+bx+c$$

where a, b, c, and d are constants and x is a variable. The study of such functions and of the rates at which they vary is of the greatest importance and it is of such elementary character that it can easily be made by the high school pupil. An elementary knowledge of these functions is of such increasing importance that the teacher of algebra should keep it definitely in mind as one of the objects to be attained.

Of course it is only the simplest problems that can be studied in a functional way. Most problems depend upon too many variables to admit study in a functional way. For example, the value of a farm depends upon, or is a function of,

the fertility of the soil, the character of the surface, the distance from a market, the character of the road to the market, the state of repair of buildings and fences, the character of the surrounding population, and many other things that are not capable of exact mathematical expression. But even in this case the functional way of thinking, which after all is the important thing, is of great value. In the words of one of the German commissions, "the power of viewing the world of phenomena surrounding us should be developed as highly as possible." The graphical study of such functions as can be studied in an elementary way is of the utmost importance. This point will be further elaborated under the head of graphical representation. Indeed most that will be said under this head may be taken as an elaboration of the function idea.

In the study of geometry the insistence upon form of expression should be continued. Here the teacher has even a better chance, for the figures relieve geometry of the severely abstract character that belongs to a large part of algebra. But in addition to the training in form and in logical processes one of the great ends to be attained by the study of geometry is a knowledge of the simpler space forms and a cultivation of the space intuition. Much has already been done in this direction in the drawing of the primary and grammar grades, but there is no reason why it should stop there. It is when he comes to study geometry that the pupil comes to know for the first time what lines and planes and geometrical figures really are. For the purpose of cultivating this space intuition, drawing is an almost indispensable adjunct to geometry. It may be called mechanical drawing, or geometrical drawing, but whatever the name may be, it should be taught as a study in form and not as a technical study. As an aid to the cultivation of the space intuition solid geometry is probably the most valuable study in the whole high school curriculum.

Finally, the teacher of mathematics should never lose sight of the fact that the education of the secondary schools should not be for the special needs of pupils who wish to take up any particular trade or profession. Some will be lawyers, some housewives, some engineers, some scientists, some day laborers, and the instruction should be so arranged as to meet the common needs of all these classes. Here and there, there *may* be

communities so thoroughly devoted to the pursuit of a single industry that it is expedient to make the high school subservient to the needs of the special industry, but such cases are exceedingly rare if they exist at all. The German Society for the Advancement of Mathematical Instruction expresses the matter well when it says: "In the secondary schools, mathematics should be a part of general culture and not contributory to technical training of any sort." The French plan of study and program of secondary instruction puts it in general terms by saying that "education should be directed first of all toward supplying the needs of the man and the citizen." No more should the secondary instruction be planned specially to meet the demands of college entrance requirements. This view of the matter does not preclude the application of mathematics to other sciences and to problems that will be likely to enter into the future life of the student, nor does it make it impossible for a well trained pupil to meet any reasonable requirement of any college entrance board.

II

THE PREPARATION FOR HIGH SCHOOL MATHEMATICS

It is extremely difficult under Wisconsin conditions to secure anything like uniformity of preparation for high school work, and this is all the more true where pupils come into the high schools in considerable numbers from the surrounding country. This being the case, the teacher must adapt the work to the pupils whose preparation is poorest. In any case the teacher who takes charge of a class of first year algebra pupils for the first time is likely to find some very great deficiencies and to feel that the work of the grade teacher has been poorly done. But where the work of the grades has been well done (and this is the rule rather than the exception) one of the most common mistakes of the high school teacher is to take too much for granted. Much of the work upon which the teacher would like to build was not clearly understood in the earlier work and much more of it has been forgotten.

There is, however, a minimum below which the high school

teacher ought not to be asked to go. The pupil ought to have a fairly good knowledge of the fundamental operations of arithmetic as applied to integers and fractions, a good knowledge of the factors of the first one hundred and fifty or two hundred integers together with a notion of the highest common factor and lowest common multiple in cases where the numbers concerned are easily factored; he ought to know expressions for the area of a rectangle, for the interest of a given sum at a given rate for a given time, for the distance traveled by a body moving at a constant velocity for a given time. If he has been properly taught he will know these results whether the data be given in numbers or literal expressions that stand for numbers. A beginning will have been made in ratio and proportion and the rule for the extraction of square root of perfect squares ought to be known. It is useless to present the *theory* of square root before the pupil reaches the high school; but he should be able to do the necessary computation with a fair degree of accuracy. Skill in numerical computation is, however, a relatively rare accomplishment and, like skill in piano playing, depends largely upon constant and prolonged practice. The high school teacher has no right to expect unusual proficiency in this line, much less to demand it.

The time given to the work just outlined ought to be very much shortened. Much progress has been made in the last two decades in the way of eliminating useless material from the arithmetics, but there is yet room for improvement. A greater insistence on the properties of the first two hundred integral numbers, a beginning toward generalization of processes, and some work looking toward the study of geometry at a later date might well replace much that should be taught as commercial arithmetic to the few special students who expect to go into banking or stock broking, a part of the work that is usually given under the head of denominate numbers, and all of cube root. It is wrong to compel the child to learn three kinds of measures of weight, and as many kinds of measures of capacity besides all sorts of foreign money when there are so many things of greater importance that are easier to learn.

It will not be out of place in this connection to speak of the desirability of some preliminary study of geometry before the

student enters the high school. For many years the best teachers have been advocating the introduction of such work. The subcommittee acting under the authority of the Committee of Ten of the National Educational Association urges it strongly in the report published in 1893. The French boy begins the study of geometry in an elementary way even earlier than he begins the study of algebra. In the German secondary schools, plane geometry is studied along with decimal fractions and the theory proportion in arithmetic. Many schools in this country have tried the plan with excellent success. A special syllabus on "inventional geometry" has been prepared for use in the grammar schools of the city of New York, and a number of excellent manuals for class use are now on the market. (1)

Theoretically, the plan of doing a small amount of elementary geometry in the grammar grades is certainly correct, but practical school men are reluctant to introduce it, urging that no place can be found for it, and that suitable teachers cannot be found. However, the question cannot be settled in any such fashion if the work really merits a place in our courses of study. The solution of the difficulty will probably be found by incorporating it as a part of our arithmetic. Indeed some of the recent text-book writers have already done much in this direction. There is no doubt but that the pupils who are not able to go on to the high school would be greatly benefited by such a program and that the efficiency of the high school work would be materially increased.

III

THE FIRST COURSE IN ALGEBRA

In most of the high schools of this country algebra is taken up as a full study in the first year. While it is by no means certain that the plan of devoting five hours a week to the work at the very outset is the best one, it will be assumed in what

(1) Among these may be mentioned,

Hanus, P., *Geometry in the Grammar Schools*, p. 52. Boston. Ginn and Company.

Failor, I. N., *Inventional Geometry*, p. 83. New York, The Century Company.

Wentworth and Hill, *First Steps in Geometry*, p. 156. Boston. Ginn and Company.

Hall and Stevens, *Lessons in Experimental and Practical Geometry*, p. 94. New York. The Macmillan Company.

follows that such is the usual practice. The primary questions before the teacher are then, what topics shall be selected and how can they be presented in the most effective manner?

As already indicated, the pupil should have made some progress in literal arithmetic when he comes into the high school, but it would be a serious mistake on the part of the teacher to assume that he is ready to take up the more formal side of algebra. In the beginning the teacher ought to explain carefully such simple problems as the finding of the distance when the rate of motion supposed to be uniform and the time are given using the formula

$$d=rt;$$

or given the dimensions of a rectangle to find the symbolic expression for the area; or given the principal, the rate and the time to find the symbolic expression for the interest and the amount. The pupil may not know that rt stands for the product of two numbers. The explanations must be repeated again and again until they are comprehended by the slowest members of the class. Indeed a good rule to go by in the early part of the work is to go about as slowly as one can go without developing habits of indolence on the part of the class.

The wise teacher will introduce the equation at the very earliest moment and will in some way or other keep it before the class almost constantly. Problems involving such equations as

$$d=rt, \quad I=prt, \quad \text{area}=ab,$$

should be solved in considerable numbers even before the four fundamental operations are taken up. Such problems should always be stated in such a way that the known things are numbers. As for example, a field whose length is l is 32 rods wide and its area is 3200 square rods. What is the numerical value of the length l? Such problems in the hands of a teacher who thoroughly understands the business in hand will not seem trivial to the pupils.

The Subject-Matter of the Course

During their first year, an ordinary class should be able to do in a reasonably thorough fashion the following topics: [1]

[1] The order of the topics is, of course, subject to considerable variation.

1. The four fundamental operations as applied to the simpler algebraic expressions together with the use of parentheses.

2. The solution of simple equations with one unknown.

3. The solution of simple equations with two and three unknowns.

4. The products of polynomials.

5. Division by a polynomial, simple cases.

6. The factoring of quadratic expressions, and of polynomials coming under one of the forms x^2-y^2, x^3+y^3, and x^3-y^3.

7. Common divisors and common multiples.

8. Fractional expressions and their combination by means of the four fundamental operations.

9. Graphical representation of functions of the form $ax+b$, assuming that the graph is a straight line.

10. The solution of quadratic equations with one unknown.

11. The theory of positive integral exponents, with exercises in fractional, negative and zero exponents *without* the theory.

12. Simple radical expressions and approximate square roots.

13. Ratio and proportion (simplest possible treatment).

14. Algebraic inequalities.

To carry out this program in the first year of the high school work, it will be necessary to cut the work down to the essentials. The two last named topics could possibly be postponed to a later period, but where pupils are allowed to drop algebra at the end of a single year, it does not seem reasonable that they should not be required to obtain some knowledge of quadratic equations.

The Four Fundamental Operations

How shall the four fundamental operations, addition, subtraction, multiplication and division, be presented most effectively to beginners? To the novice, there is no more important question and no question upon which the books give such varied and unsatisfactory information. The reasons are probably not far to seek. During the last three or four decades a very great amount of closely critical work has been

done upon the foundations of mathematics and the writers of elementary books have endeavored to meet the supposed demand for rigor of presentation that has grown out of this critical study. In many cases the result has not been altogether fortunate. The presentation is neither scientific nor pedagogical.

A thoroughly scientific treatment of algebra would begin with adequate definitions of number and of the four fundamental operations and, proceeding upon this basis, would make every part of the logical structure absolutely secure. The problem that presents itself to the teacher of elementary mathematics is totally different from that considered by the mathematician who seeks to place the foundations of algebra on a solid basis. The treatment of the fundamental operations to be presented to a child of thirteen should be scientific so far as possible, but it *must* be pedagogical, if it is to be in any way effective. It is necessary that the teacher should know both aspects of the subject.

In the present section, an attempt is made to make the presentation as simple as possible without departing far from scientific accuracy and at the same time to point the way for the teacher who may wish to make a more careful study of the foundations of algebra.

The pupils entering the high school are already familiar with the fundamental operations so far as their application to combinations of integral and fractional numbers is concerned. However, these operations possessed an essentially different character in that the numbers combined lost their identity in the result. For example, the product of 3 by 8 is 24. The result taken by itself gives no intimation of the identity of the original factors. They might have been 4 and 6 or 2 and 12 or even 1 and 24. On the contrary, in the algebra the combinations are for the most part merely formal. The product of a by b is ab and there the matter ends. We do not even inquire into the meaning of the factors. It is this formal character to which algebra owes its simplicity and to which also many of the difficulties that enter may be traced. In the arithmetic, the operations were not sharply defined and no note was made of the fact that one definition is made when the two numbers to be combined are integers and another

when they are fractions. Irrational numbers were not considered at all.

The four fundamental operations of algebra are, as already indicated, addition, subtraction, multiplication, and division. To these four might be added two others, involution and evolution, which are not, however, independent operations when considered in their simplest aspects. These four operations are at the outset defined for positive integers alone. By successive steps, they are made to apply to negative numbers, to fractional numbers, to imaginary numbers, and to irrational numbers. These extensions are accomplished *through the definitions of the new numbers themselves.*

The operations are always and everywhere subject to three laws, viz.:

1. The Commutative Law.
2. The Associative Law.
3. The Distributive Law.

These three laws which serve to define the subject of algebra, since every step is taken with a recognition of their validity, may be stated symbolically as follows:

The Commutative Law for addition asserts that $a+b=b+a$, and for multiplication it asserts that $ab=ba$.

The Associative Law for addition is $(a+b)+c=a+(b+c)$, and for multiplication it is $(ab)c=a(bc)$.

The Distributive Law for multiplication is $a(b+c)=ab+ac$.

On the scientific side, two lines of procedure are possible. In the first, one defines completely number and the four fundamental operations and proves the three laws as theorems. In the second, one defines positive integral number and the four fundamental operations for positive integral numbers and then assumes that the three laws hold for all extensions that may be made to the number system. [2] Either method involves very serious difficulties and is wholly out of the question for high school pupils.

Good pedagogy requires that the beginning of algebra be

[2] The fundamental difficulty lies in our definition of number. The whole matter is relatively simple so long as we deal with positive numbers whether they be integral or fractional, but the difficulties begin when we introduce negative numbers and imaginaries and culminate when we attempt to get an adequate theory for irrational numbers. References will be given below for the use of those teachers who wish to go deeper into the difficulties involved.

made as simple as possible. To this end it is advisable that in elementary instruction *we should both assume the three fundamental laws and define the four fundamental operations for all kinds of number.* Not only should we do this but it is not necessary even to state the fundamental laws for beginners. Pupils will accept the commutative law and the associative law without question and they will be familiar with the distributive law by the time they have had a good drill in multiplication. But the teacher should know that it is *impossible* to prove that $a(b+c)=ab+ac$. The thing may, and should be *illustrated* by numerical examples, as $2\times(3+4)=14$ and $2\times3+2\times4=14$; also $\frac{1}{2}\times(6+8)=7$ and $\frac{1}{2}\times6+\frac{1}{2}\times8=7$. If we use other numbers, as $\sqrt{2}(2+\sqrt{3})$, verification is impossible for the present, and for the great majority of pupils it will always remain so. If we consistently adhere to this principle of assuming more than is absolutely necessary when the proofs of the things assumed are beyond the reach of the pupils, many simplifications will follow as we shall find later on.

The most serious difficulty that confronts the teacher of first year pupils in the early stages of the work arises in connection with negative numbers. Not only is the idea of a negative number in itself difficult for the pupil to grasp but the laws by which they are combined are most mysterious to him; and it must be admitted that some of the text-books in current use do not give him much assistance.

The first requisite to successful work at this point, as everywhere, is absolutely clear thinking on the part of the teacher. Most teachers and most text-books attempt to prove too much. When once the teacher realizes that negative numbers are *new* numbers, the result of definition, and that sums and products of such numbers must be *defined*, the matter will be greatly simplified. Many illustrations will occur to the teacher. The thermometer scale will probably be the simplest, all things considered. The debit and credit side of an account, north and south latitude, and other examples are utilized in all the books. But all these things can only serve to illustrate the *need* of negative as opposed to positive numbers and in reality throw little light upon the nature of the

numbers themselves. The devices that have been invented to show how to determine the sign of the product of two numbers of which one or both is negative usually assume too much on the part of the student, and in many cases really obscure the issue by bringing in ideas that are more complicated than the notion to be explained. It is absurd, for example, to attempt to explain the law of the signs by the balance when the pupil has no notion either of a force or of a moment.

The simplest and also the best method for beginners is comprehended in the rules frequently given, viz.:

I. *To subtract a negative number is to change its sign and add.*

II. *The product of two numbers having like signs is a positive number and the product of two numbers having unlike signs is a negative number.*

These rules are concise and easy to handle and they have the further advantage that they come about as near to the real reason underlying the rules as the pupil is likely to get during his high school course whatever method his teacher may employ. They are scientific because they are equivalent to definitions of addition and multiplication with negative numbers.

Of course one may reach the same end by saying that $a\times(-b)$ is a negative number, b, taken as many times as there are units in the multiplier a, i. e., we add the negative number $-b$, a times. The result is obviously $-ab$. Next we should have $(-a)\times b=b\times(-a)=-ba=-ab$ by the foregoing. Finally we would have $(-a)\ (-b)=-[a\times(-b)]=-[ab]=+ab$. But you have employed a process that not one pupil in ten will be able to reproduce with any real comprehension of the meaning and which after all applies *only in case a and b* are integers. It is much simpler to make some such assumptions as are made in the rules given above. [3]

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[3] Professor Fine formulates the definitions of the fundamental operations for integral numbers admirably as follows:

Definitions of addition and subtraction.

1) $a+b$ is to mean the bth number after a.
2) $a-b$ is to mean the bth number before a.
3) $a+0$ and $a-0$ are to mean the same number as a.
4) $a+(-b)$ is to mean the same number as $a-b$.
5) $a-(-b)$ is to mean the same number as $a+b$.

Definition of multiplication.

1) $0.a$ and $a.0$ are to mean 0.
2) $a(-b)$ and $(-a)b$ are to mean $-ab$.

It is scarcely necessary to go into detail concerning further work in the four fundamental operations. The pupil should be able at an early period to perform simple multiplications with monomials and binomials, to have clearly in mind the definition of a square and a cube written in the usual notation, to recognize the fact that $a^2 \times a^3 = a^5$ not through a formal demonstration but by reason of the fact that he knows he has before him five factors, to perform any division where the divisor is a monomial and easy divisions where the divisor is a binomial, and finally to recognize the truth of such statements as $a^3 \div a^2 = a$ just as he knows corresponding facts in multiplication where integral exponents are concerned. Careful study will have been made of the use of parentheses, but it is not necessary to make the students go through a large number of complicated examples. The principle is sufficiently well illustrated by examples which have negative numbers within a single parenthesis preceded by negative sign and there is little need of going to two sets of parentheses.

The Beginning of Graphical Representation

Graphical representation in algebra does not often add much to our knowledge, but it makes certain facts stand out vividly and gives us a grasp of them that would be hard to obtain otherwise. For this reason it is a valuable aid to the teacher and to the elementary pupil as well as to the more advanced student. Moreover it is used in so many different connections that no student can afford to be ignorant of at least the elements of the subject. The pupil who is to make any real use of graphical representation needs to begin the

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3) $(-a)(-b)$ is to mean ab.

Similar definitions must of course be employed after one has defined fractional and irrational numbers. In his *Theory of Functions of a Real Variable*, pp. 16-18, Hobson, approaches the matter from another standpoint, virtually making the three fundamental laws the basis for his definition of negative numbers.

The point to be made here is that, owing to the intrinsic difficulty of either method, the teacher will both assume the fundamental laws and define the fundamental operations. Such procedure is sanctioned by the best mathematical pedagogy which recognizes the wisdom of admitting a larger body of assumptions than is necessary to furnish a basis for the subject.

For further information regarding the fundamental operations the teacher should consult 1) Fine, *College Algebra*, pp. 19-20; 2) Weber-Wellstein, *Encyklopädie der Elementar-Mathematik*, Vol. I, § § 12, 13, where one will find definitions for addition, subtraction and multiplication of negative numbers.

subject early and to return to it again and again during his course. Fortunately the beginning is easy and of such nature as to assist the pupil materially in getting a firm hold upon the elements of algebra.

Graphical representation had its origin in the idea of measurement. The measuring stick is a straight line with divisions marked upon it such that any one division is equal to any other and these divisions are always numbered in order from a given point, viz., the end of the stick. But it is not necessary that a given line should always represent a length. It may be used to represent a time, or a number of dollars, or a weight, in which case a line twice as long would represent a time, or a number of dollars, or a weight twice as great, and similar statements would be true for a line of any other given length. A considerable number of easy examples will serve to make the matter clear. Such examples as the following should be given in abundance:

1. A man travels 4 miles west and then 5 miles west. How far and in what direction is he from the starting point?

2. If he travels 6 miles west and then 2 miles east, how far and in what direction is he from the starting point?

3. If he travels 6 miles west and 10 miles east, where is he relative to the starting point?

4. A man starting out with no money gained $300 and then spent $700. What was the state of his finances at the end?

The teacher must see that the graphical representation proceeds exactly as fast as the explanation and must not rest until the pupils can "chalk and talk" at the same time in explaining similar problems. Graphical addition and graphical subtraction should be made thoroughly familiar even at this early period. When the pupil sees that a change of sign is always accompanied by a reversal in direction many things in the theory of negative numbers will be easier. For example, $2-(-3)$ may be interpreted that a man started from a given point and traveled two miles in a certain direction, then reversed his direction twice and traveled three miles further. Such considerations lead to the notion of a "complete scale" made up of positive and negative integers and zero, which may

be arranged at equal intervals and in a certain order along a straight line, as in the figure.

$$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$

Bear in mind that the scale consists of the numbers and not the straight line with its points of division. Given any two numbers in this scale of which zero may be one, that one is greater whose position on the line is farther to the right. Fractional numbers will be inserted between the integers.

The important things to be noted are:

1. Magnitudes of any kind or even abstract numbers may be represented by lengths measured from a given point on a straight line.
2. The unit of length is arbitrary.
3. Any point on the line may be taken as the starting (zero) point.
4. Numbers representing lengths measured in one direction from the starting point have one significance (one sign), while numbers representing lengths measured in the opposite direction have a different significance (a different sign).

This may all seem very elementary to the teacher who has studied analytical geometry, but it is wholly new to the beginner in the high school, or, if it is not new, it will need to be called to mind again.

Equations of the First Degree

Before taking up the discussion of linear equations, a word should be said regarding form. In one sense algebra may be looked upon as a sort of language which may be used as a medium of expression for facts that could not be expressed in ordinary language except with great difficulty or at great length, if at all. An equation itself is a complete sentence which has a definite place among other sentences which form a paragraph. It would perhaps be carrying matters rather too far to insist on all the niceties of punctuation and paragraphing that are required in the classes in English, but something can and ought to be done in this direction. The teacher should insist on the best form that is attainable and should refuse to accept written work that is not as neat as the pupil

can make it. Even more emphasis should be placed upon oral expression. Such slovenly and incorrect expressions as "the equation equals to" and others equally bad should never be allowed to pass without the criticism they deserve. The habit of careful and accurate expression is an invaluable thing to any pupil. He should be taught to place his equations in proper order, to read them correctly and with an understanding of their meaning, to translate the mathematical language into clear and concise English and to clothe in the simplest mathematical language the problem that he finds expressed in ordinary English.

The high school teacher of the present day may assume that the first year pupil has already some acquaintance with the equation. Seventh and eighth grade pupils are now required to do a considerable amount of work in the way of the solution of equations of the first degree with one unknown having only numerical coefficients. The teacher of algebra will not, however, make the fatal mistake of assuming that the knowledge acquired in the arithmetic is all available. Some of it was not clearly understood at the time; a considerable part of it was necessarily done in a mechanical fashion; and in any case most of it has been forgotten.

The equation must then be taken up from the beginning, in a more general form to be sure, but with a clear and adequate explanation at every point. With the Euclidean axioms that "if equals be added to equals the sums are equal," and "if equals be subtracted from equals the remainders are equal," it is easy to teach the pupil exactly why he changes the sign when he transposes a number from one side of an equation to the other.

The matter of a standard form for equations may receive some attention at an early period. Transposition and collection of terms in a simple equation of first degree lead always to one of the two forms $ax+b=0$ or to $ax=b$, of which the first form is preferable. Similiarly, one may insist that all systems of simultaneous linear equations be brought to one of the two standard forms

$$\begin{aligned} ax+by+m&=0 \\ cx+dy+n&=0, \end{aligned} \qquad \text{or} \qquad \begin{aligned} ax+by&=m \\ cx+dy&=n, \end{aligned}$$

as must be done before they can be solved conveniently.

Two methods of elimination, viz., the method by addition and subtraction and the method by substitution are sufficient. The addition and subtraction method is useful because it is so easy to apply, while the method by substitution should be emphasized because it applies to any pair of equations of which one is of the first degree. Before the class leaves the subject the pupils, should be able to solve any system of two or three equations of the first degree and should be able to put the results of such a system as

$$ax+by=m$$
$$cx+dy=n$$

in the form

$$x=\frac{md-nb}{ad-bc}, \quad y=\frac{na-mc}{ad-bc},$$

though it is not advisable at this stage to use the formulas as a means of solution unless the solution is put in determinant form.

If the determinant form

$$x=\frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}, \qquad y=+\frac{\begin{vmatrix} a & m \\ b & n \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}},$$

is used, as it may be with perfect propriety, no attempt should be made to define a determinant. One should say simply that the determinants are nothing but other ways of writing the differences, *ad—bc*, etc., occurring in the solution.

This topic will not present serious difficulties to the wide-awake teacher.

In the early work at least it is advisable that all work be tested in some way or other, and it is most important that the pupil understand that a root of a single numerical equation is something which when substituted for the unknown number renders the equation numerically (identically) true. The pupil may do his own testing, not because there is any peculiar merit in the testing process but because this is probably the best way to compel accuracy on his part and to make him feel sure of himself. But it must always be understood by the pupil as well as by the teacher that this testing process is subsidiary to the main result and that the work is not done merely for the purpose of securing answers.

The Second Stage of Graphical Representation

In the first step in graphical representation, the fundamental notion was that of a magnitude that may take different values which may be represented by lengths measured along a straight line. In the second step, we have to introduce a second magnitude or function whose value depends upon the value of the first. The values which this second variable take cannot be represented on the first line without confusion so we agree for convenience to measure them along perpendiculars to the first, each perpendicular to be erected at the point corresponding to the particular value of the first variable. For example, the population of a city varies with the time measured from a given date arbitrarily chosen; the time is measured along the first or horizontal line and the population may be represented by lines drawn perpendicular to the first. This process may be called the charting of statistics or by any other convenient name, but whatever be the name by which it is called, the idea of one set of numbers which depend upon another set is the fundamental one and should be kept constantly in view.

For this work squared paper is indispensable and a section of ruled blackboard is a valuable aid. (4)

After the simple problems involving the charting of statistics, problems involving uniform motion will be found most useful. Problems like the following with variations suggested by the teacher may be used:

1) A man rides at the rate of 8 miles per hour.

2) A man starts from a point 5 miles from home and rides in the opposite direction at the rate of five miles per hour.

3) A man starts from a point 3 miles from home and travels toward home at the rate of 6 miles per hour.

Construct the graph giving the distance from home at a given time for each case.

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(4) Squared paper can now be obtained of any dealer in school supplies. The best paper contains at least two different kinds of rulings, one of which has ten divisions corresponding to one of the other. With such paper one can make computations very accurately to tenths of the original unit.

To rule a section of blackboard lay it off in two inch squares with ordinary white paint, using a brush that makes a mark not more than a quarter of an inch in width. If the board is of slate, the ruling can be erased at any time by washing with kerosene and the board can be restored to its original state by washing out the kerosene with gasoline.

It is easy to guess that the graph is a straight line in each of the examples given. Assuming such to be the case, the line is easy to construct when one has any two points on it. The graph then serves as a table of values from which the distance from home at any given time may be read *without* computation.

Under each of the three types of problems above the teacher should give a sufficient number of exercises to bring out the fact that *the faster the man travels the steeper the graph will be.*

In the same way the formula for simple interest,

$$I=prt,$$

may be exhibited in graphical form for any given principal and rate, by taking t as the variable. If the principal and rate be constant the graph showing the interest will again be a straight line. A higher rate or a larger principal will result in a steeper line, but it is always a straight line. The line once drawn, all problems in interest involving a given rate and a given principal are solved by a simple inspection of the graph. The graph takes the place of an interest table. A still more useful exercise would be to take an ordinary rate and a given time, say 90 days, and allow the principal to vary.

All the foregoing graphs are graphs of expressions or functions of the form

$$ax+b.$$

This expression, whatever values a and b may have, has a straight line graph. It will not be difficult for bright boys to find the significance of the coefficients a and b and of the quotient $\frac{a}{b}$.

The pupil will have made a long step when he knows that the graph of $ax+b$ is *always* a straight line and that the line will give him by simple inspection the value of the expression for all values of x i. e., it is a graphical representation of the function $ax+b$. He does not have to prove the fact that the line is straight in order to know it. The proof is out of the question until he knows something of similar triangles.

Harder Problems in Multiplication and Division

In the multiplication of polynomials, the books have examples enough and to spare, but not many of them lay proper emphasis on the important forms that are to be used in later work. The pupil ought to be able to work out the product of any two ordinary polynomials of which one is a binomial or a trinomial, but there are certain type forms that ought to be so thoroughly ground in that the results can be given orally without any hesitation. These type forms correspond, of course, to the factorable forms that are introduced later, but if they are brought to the attention of the student in an emphatic fashion as early as possible there will be much less trouble with the factoring. The student who has been well taught should not hesitate when asked to give such identites as

1. $(a+b)^2=a^2+2ab+b^2$
2. $(a-b)^2=a^2-2ab+b^2$
3. $(x+a)(x+b)=x^2+(a+b)x+ab$ with all possible combinations of sign for a and b.
4. $(a+b)(c+d)=ac+ad+bc+bd$
5. $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$
6. $(a-b)(a^2+ab+b^2)=a^3-b^3$
7. $(a+b)(a^2-ab+b^2)=a^3+b^3$
8. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

In addition to these the identities

9. $(a-b)^2+4ab=(a+b)^2$
10. $(a+b)^2-4ab=(a-b)^2$

will be found most useful. Very many examples in factoring, indeed practically all the high school pupil will ever need to solve unless he takes up mathematics beyond the high school, can be solved by means of these forms.

Pupils should be drilled in finding the results of such numerical examples as $(1.04)^2$ by (1); $(98)^2=(100-2)^2$ by (2); $5\frac{1}{2}\times7\frac{3}{8}$ by (4). At first pencil and paper may be used in these computations but oral should be gradually substituted for written work and finally should be used altogether.

Much has been said regarding the verification of algebraic

identities by substituting numbers. For example, if $a=5$ and $b=3$ we have

$$(a-b)(a^2+ab+b^2)=98 \text{ and also}$$
$$a^3-b^3=98$$

Similarly if $a=3$ and $b=4$

$$(a-b)(a^2+ab+b^2)=-37 \text{ and}$$
$$a^3-b^3=-37$$

Such work is valuable as showing the difference in character between an identity and an equation, and for that reason some of it may be done. On the other hand, it is difficult to see just what is to be gained when such work is to be done for the purpose of checking results, for the true spirit of algebra is that of ever-widening generalization. It is the numerical side of the work from which the pupil needs to free himself as rapidly as possible. The pupil who insists upon verifying by actual multiplication and division every problem that he solves by logarithms will make no progress. Moreover, the fact ought also to be recognized that by far the greater part of the really important results in algebra are of such a nature that such checks are impossible.

The work in division will present no very great difficulty if care is taken to present only simple problems, and as a matter of fact, it is very rarely that any other sort of problems in division occurs in elementary mathematics.

Factoring

The subject of factoring is one of the most important in elementary algebra, not for its own sake but for the sake of its applications. It is not, however, necessary to insist on a very large number of complicated examples which serve rather to impress the pupil with the difficulty of the subject than to help him to understand it. It would be better, rather, to give a large number of easier problems that will make him understand that the number of cases is not great and that all problems can be referred to the few standard cases. The teacher should understand, however, that pupils have not been drilled with sufficient thoroughness if they have to spend much time in referring a given example to the appropriate case.

The number of type forms is after all relatively small, and by far the larger number of examples comes under not more

than four of these. It is assumed, of course, that monomial factors will receive proper attention, though there should be no difficulty with such. The following forms given in the order of their importance represent all that are used unless the computer employs special methods:

1) $a^2-b^2=(a+b)(a-b)$
2) $x^2+(a+b)x+ab=(x+a)(x+b)$
3) $am+an+bm+bn=(a+b)(m+n)$
4) $a^3-b^3=(a-b)(a^2+ab+b^2)$
5) $a^3+b^3=(a+b)(a^2-ab+b^2)$
6) $mnx^2+(anx+bmx)+ab=(mx+a)(nx+b)$

The great majority of problems will be solved by means of 1) or 2). However, one must note that various combinations of sign may enter in form 2). Of course the teacher will see that the class has sufficient drill in problems in which the form differs materially from the type form given above. For example, $(x+3)^2-(x-4)^2$, x^6-y^6 and others will be easily within reach of the class. The factoring of any quadratic expression will be discussed in the outline for the third half year's work.

Finally, the teacher must remember that the great majority of the pupils will never have any occasion to do a complicated exercise in factoring and that the examples should be no more difficult than are necessary to give the pupil facility in computation. This facility will never be reached by giving a problem that the pupil cannot solve. It should be strongly impressed upon the pupils that when an expression is once factored it should remain in that form unless it is absolutely necessary to multiply out again.

When the easier cases of factoring have been mastered, it is advisable to take up the subject of roots in a very elementary way. The definition of a root and the radical notation may be introduced. Then problems in considerable number like $\sqrt{a^6b^4c^{10}}$ may be brought in to be followed by the extraction of the roots of integers which are perfect squares. The pupil who sees clearly that the processes employed in extracting the square roots of $a^6b^4c^{10}$ and 1764 are identical has made a good step in the line of progress. The approximate extraction of square roots should be delayed until a later period. Cube root extraction by any method than factoring may be

omitted altogether without serious detriment to a high school course.

Highest Common Factor, Lowest Common Multiple and Fractions

Little need be said concerning the first two subjects except that any method of attack for either other than the method of factoring is for first year pupils practically useless. Indeed both subjects should be taught essentially as parts of factoring.

The subject of fractions may be cut down to the merest essentials. The teacher will endeavor to see that pupils grasp the essential notions without having the main issues obscured by difficult examples. Exercises which carry the pupil back to his knowledge of arithmetical fractions will be most effective to begin with. If the pupil can tell quickly what to do when asked to add, subtract, multiply or divide such fractions as $\frac{a}{b}$ and $\frac{c}{d}$, the main part of the struggle is won. A fairly large list of easy examples will do more to fix principles in the mind than the more complicated ones the like of which he will probably never see in actual work.

Quadratic Equations

The solution of quadratic equations, which can be advantageously taken up before the topics mentioned on page 34 will be most easily introduced through the method of factoring. This method which is the most direct, has some advantages for later work also. Care must be taken, however, to be sure that the pupils know that a product is zero when one factor is zero and conversely. The teacher will find that it is much easier to make the class believe that this is true than to get them to understand any of the so-called proofs of the fact. The main thing is to get them to consent to the fact.[5] In this way the student is placed in immediate posses-

[5] The best writers make the relations $a \cdot 0 = 0 . a = 0$ a definition. (See footnote to p. 18 above.) Assuming this definition, it is easy to prove that ab cannot be zero unless $a = 0$ *or* $b = 0$. but the proof is not suitable for high school pupils. The best the high school teacher can do is to appeal to the intuitions of the class through numerical examples. The product of any two definite non-zero numbers as 6 and 4 or 10 and .001 is not zero, while any number of zeros is zero. The wise teacher will present just as little of the matter as may be necessary to secure acceptance of the fact on the part of the class.

sion of both roots and the difficulty that frequently arises with such special forms as $ax^2+bx=0$, $ax^2=0$, is very much minimized. The pupil must understand from the beginning that the solution is not complete until *two* roots are found.

While the method by factoring is universally applicable, it is less convenient in actual practice when one has to deal with such equations as

$$7x^2+5x-37=0.$$

Even in such cases the pupil should be made to understand that the factoring can be done by first removing the coefficient of x^2. In the general case one has

$$ax^2+bx+c=a(x^2+\frac{b}{a}x+\frac{c}{a})=0$$

and the last equation is equivalent to

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

which has the same form as the equations that have been solved by factoring.

When the class has acquired some facility in handling easy quadratics by the method of factoring, one other method may be taken up. The notion of a standard form already mentioned in the case of linear equations should be utilized here. All things considered the form $x^2+px+q=0$ is probably the best form into which one may throw a quadratic equation, but there are others, such as $ax^2+bx+c=0$, $ax^2+bx=c$, $x^2+bx=c$, which are in common use. The form that is chosen is a matter of little consequence, but some form should be selected and then it should be made clear that all quadratic equations can be thrown into the chosen form. From this standard form a standard solution may be derived, which, however, is not necessarily a solution for a different standard form.

There is no best method for finding the solution of the quadratic equation and it matters little which of the so-called methods is adopted provided that method is fully understood If one has chosen as the standard form

$$x^2+bx+c=0,$$

the solution is easily connected with previous work if the process be carried out somewhat as follows:

$$\begin{aligned} x^2+bx+c &= x^2+bx+\frac{b^2}{4}-\left(\sqrt{\frac{b^2}{4}-c}\right)^2 \\ &= \left(x+\frac{b}{2}\right)^2-\left(\sqrt{\frac{b^2}{4}-c}\right)^2 \\ &= 0. \end{aligned}$$

One has here a form which may be factored at once, or the term containing the square of the radical expression may be transposed to the other side and the square root of both sides of the resulting equation may be extracted. It will take a good deal of patience to get every member of the class to understand the process of completing the square term by adding $\frac{b^2}{4}$ to x^2+bx. The explanation will be done by means of the equations with numerical coefficients and not until after the class is thoroughly familiar with such equations will the teacher ask for the solution of any equation with literal coefficients. When the equation has irrational roots, one should insist on the retention of the radical form rather than require the pupil to work out the approximate square root.

There is no valid reason why the solution of the simultaneous system consisting of a linear and a quadratic equation might not be taken up at this point. The pupil who can solve a linear equation and a quadratic has to take but a single additional step, and that not a new one, to get the solution of such a system as

$$2x-3y=7.$$
$$x^2-5x+6y=13.$$

Equations and Identities

Something has already been said concerning identities. It is none too early to try to make clear the difference between an identity and a conditional equation. The identity is true for every set of values that can be given to the letters occurring in them. One may substitute for a and b in the identity

$$a^2-b^2=(a+b)(a-b)$$

any pair of values whatever and the result is the same on both sides; every possible value of x but one will satisfy

$$\frac{x^2-5x+6}{x-2}=x+3.$$

(If the pupil happens to stumble upon the exceptional value and finds a meaningless expression no harm will be done.) On the other hand, there is in the totality of numbers but a single one which will satisfy the equation

$$\frac{x^2-5x+6}{x-2}=0.$$

Again there are infinitely many pairs of values of x and y which will satisfy the conditional equation

$$2x+3y+4=0$$

though any pair taken at random as $x=2$, $y=1$ will not in general satisfy it.

The difference between the two sorts of equalities is a fundamental one. Indeed, if both sorts are to be called equations, the latter may by way of distinction be called "conditional" equations since the truth of the equality depends on the substitution of certain numbers for x and y. All our transformations are made by means of identities, while all our problems are stated in terms of equations. Whenever one comes upon an identity in the solution of a problem, it is certain evidence that some mistake has been committed or that the original data were not adequate. It must not be thought, however, that for this reason the identity is of no use. All the equalities formed in our chapters on factoring and fractions are identities.

The characteristic properties of identities, their use and the methods by which they are detected are subjects beyond the high school pupil, but he ought by all means to be made to understand the difference between the identical equation and the conditional equation.

Inequalities

An elementary knowledge of inequalities will be found to be of great use. In numerical inequalities, no note is taken of the signs of the quantities compared and the comparison of two quantities having different or negative signs would be im-

possible, unless we consider numerical values alone. For example the inequality $2<4$ is numerical. 2 is numerically less than 4 but algebraically greater than -4. Again -2 is algebraically greater than -4. To define an algebraical inequality, we say that a is greater than b when $a-b$ is a positive number and a is less than b when $a-b$ is a negative number. According to this definition -2 is greater than -4 since $-2-(-4)=+2$.

In explaining algebraic inequalities, free use ought to be made of the complete scale. Indeed the teacher may define an algebraic inequality with reference to a scale, i. e., one number is greater than another when the point which represents it is farther to the right in the line on which the scale [6] is represented. This definition will lessen the difficulty that arises when the pupil comes to inequalities between negative numbers. He ought to have no difficulty, for example, in seeing that -2 is greater than -4.

It is not at all necessary to prove an elaborate set of propositions. One may assume as axioms, as Euclid does, the propositions:

1) If equals be added to equals the sums will be equal,

2) If equals be taken from equals the remainders will be equal,

3) If equals be added to unequals the sums will be unequal,

4) If equals be taken from unequals the remainders will be unequal, making the further assumptions that in 3) and 4), the resulting inequalities exist in the same sense as the original inequalities. This done, it will be sufficient to prove that,

1) Any number may be transposed from one side of an equality to the other if at the same time its sign be changed.

2) If all the signs of an inequality be changed the inequality sign must be reversed.

3) All the terms of an inequality may be multiplied by a positive constant without changing the sense of the inequality.

4) An inequality may be cleared of fractions in exactly the same way as an equation is cleared of fractions.

Numerical illustrations should be given freely.

It is very desirable, though not absolutely necessary, that

[6] For a complete definition see Fine's College Algebra, Nos. 8, 18, 60, 105, 156.

this subject should be taken up as a part of the first year's work.

Radicals and Approximate Computation

In a first course in algebra, a very brief treatment of radicals is all that is necessary. From the standpoint of advanced mathematics, it is better no doubt to make the work in radicals depend upon the theory of exponents, but for the beginner a more direct method is certainly better.

All necessary work can be done by means of two principles that may be stated as follows:

$$1)\ \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

$$2)\ \sqrt{\frac{a}{b}} = \sqrt{\frac{a}{b}}$$

By assuming that two identical equations may be multiplied term by term, these principles may be easily verified. For the first, we would have the product term by term of the two equations

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

This product is evidently

$$\sqrt{a} \cdot \sqrt{b} \times \sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \cdot \sqrt{ab}$$

or

$$ab = ab$$

which is identicaly true. It will probably be just as well to assume the truth of 1) and 2) and say that satisfactory proofs may be given later.

These two principles given here in symbols should be translated into words and read both forward and backward so that they will be recognized in whatever form they may appear.

By means of the two principles one may,

1) Reduce any expression like $a\sqrt{b}$ to the form $\sqrt{a^2b}$.

2) Take a square factor from under the radical sign.

3) Perform all necessary multiplications and divisions upon radicals.

4) Reduce fractions with quadratic monomial or binomial radicals as denominators to equivalent fractions having rational denominators.[7]

This work may be done before the actual work of approximate square root extraction is taken up, but it should be explained that the end in view is almost invariably an approximate numerical result. For example, if a machinist has data given him for his work, which involves $\frac{5}{\sqrt{2}}$, he cannot use it until he has the result expressed approximately as a decimal. If he needs the approximate result, true to the fourth decimal place, he can either extract the square root of 2 to four places and divide 5 by the four-place result, or he may reduce the expression to the form $\frac{5\sqrt{2}}{2}$, where he has to multiply the four-place root by 5 and divide by 2. The difference in the labor involved is even greater in such a case as finding the approximate value of the expression $\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}$.

It will be advisable to let the pupils know that the method used in the case of quadratic surds apparently does not apply in such a case as,

$$\frac{1}{1+\sqrt[3]{2}},$$

though it is not worth while at this stage to have him determine the multiplier in this case.

If the pupils have not yet learned how to extract the approximate square root of numbers that cannot be factored, that work may be introduced at this time. The intelligent teacher will appreciate the wisdom of the French program on this point when it advises the teacher to give "the practical rule, without the theory." However, if one must give the reason, it is most easily done by taking a concrete example whose root is expressed in two figures and then asking the pupils to go ahead on the basis of the rule derived from the simple case. For example, if one has the square 2809, it is seen at once that this number lies between 2500 and 3600 and con-

[7] In the interest of strict accuracy, one ought never to allow the use of the expression "to rationalize the denominator." The rational denominator is always different from the denominator containing the radical expressions.

sequently its root lies between 50 and 60. The root is then of the form $10t+u$, where t and u are the digits in the tens and units places respectively, and the number whose root is to be extracted is of the form $100t^2+20tu+u^2$. In the example $t=5$ and $100t^2=2500$ while $20tu=100u$. It follows that $100u+u^2=2809-2500=309$. The second figure is seen to be approximately 3 since in the equation

$$100u+u^2=309$$

the term u^2 is small as compared with $100u$. The remainder of the process consists in verifying the guess that has been made, viz., that the last figure is 3. One may of course use the form $t+u$ for the root and the number is then $t^2+2tu+u^2$. The teacher will then be able to explain the correctness of the arithmetical process as ordinarily given and to extend it to larger numbers.

If one has to deal with a number which is not a perfect square, as 2978, the best that can be done is simply to say that an approximate result is obtained by carrying out the process given above by adding pairs of zeros. It is nearly useless to go into the theory with first year pupils in the high school. The results may be justified in every particular case. In the example just given one knows that

$$(54)^2 \angle 2978 \angle (55)^2.$$

Whence it follows that 54 is an approximate result, true as far as it goes. Similarly it is easy to show that

$$(54.5)^2 \angle 2978 \angle (54.6)^2$$

whence 54.5 is the result to the first decimal place. The process may be carried on as far as one pleases, but it should be clearly understood that it is a waste of time to carry results to a degree of accuracy beyond that required by the problem in hand. The pupil should know also that if he seeks the *best approximation* the last figure obtained by the ordinary process of root extraction may not be the best one to retain. In the example just given 54.6 is nearer the truth than 54.5 since the next figure is 7.

The extraction of the square root of polynomials should go hand in hand with the numerical problems and each should be used to throw light upon the other. It is easy to carry such work farther than is profitable, however, since such problems

do not often present themselves to the pupil who drops his mathematics at the end of his high school course, and only very rarely to the student of advanced mathematics. It is extremely important that the teacher should recognize the fundamental significance of the irrational numbers both in algebra and in geometry. Speaking from the point of view of algebra, one may say that irrational numbers were invented to make root extraction of positive numbers possible in all cases, just as negative numbers and fractional numbers were invented to make subtraction and division possible in all cases.

That the irrational numbers are wholly different from the rational numbers is seen when one defines a rational number as the quotient of two integers which are relatively prime. From this it follows that the square root of 2 can not be rational, for suppose it were true that

$$\sqrt{2} = \frac{m}{n}$$

where m and n are integers. Squaring both sides of the supposed equality, one finds

$$2 = \frac{m}{n} \frac{m}{n}$$

This result is absurd for if n is unity, we have a prime number 2 equal to a composite number $m\ m$; while if n is different from unity, we have the integer 2 equal to the fraction $\frac{m\ .\ m}{n\ .\ n}$ since mm and nn have no common factor.

It should be noted that such forms as $\sqrt{2}$, $\sqrt{3}$, $2+\sqrt{5}$, are merely symbols and do not in any sense define numbers. Concerning the definitions of irrational numbers and their fundamental importance in geometry, a fuller statement is given below. (See foot-note, p. 75.)

The Theory of Exponents

The definition of an integral power of a *base* a, as a^5 or a^n, where n is an integer, as the product of several factors, has already been given and pupils should know by this time that

$$a^m \cdot a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$

the latter equation having sense only when $m>n$. These re-

sults must be demonstrated anew and other results added to them. All computations in which exponents enter can be done by means of three laws, but the teacher will find it easier to use five.[8] These laws are:

$$\text{I.}\quad a^m \quad a^n = a^{m+n}$$
$$\text{II.}\quad a^m \div a^n = a^{m-n}$$
$$\text{III.}\quad (a^m)^n = a^{mn}$$
$$\text{IV.}\quad (ab)^m = a^m b^m$$
$$\text{V.}\quad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

These laws are, for integral values of m and n and with the restriction already mentioned in the case of II, theorems which may be demonstrated. They should be translated into ordinary language and should be gone over again and again, so that there is no danger that they will ever be forgotten, because they are almost as important for the purposes of algebra as the multiplication table is for arithmetic. For example, the third translated into ordinary language would read, "a power of a power of a given base is equal to that power of the same base whose exponent is the product of the exponents of the two powers." The essential features should be brought out by such questions as "when does one add exponents?" or "when does one multiply exponents?"

The whole five laws or theorems should be carefully demonstrated. To demonstrate the fourth, for example, one has

$$(ab)^m = (ab)\ (ab)\ (ab) \text{ — to } m \text{ factors (by definition)}$$
$$= (a\ .\ a\ .\ a\ . \text{ — to } m \text{ factors})\ .\ (b\ .\ b \text{ — to } m \text{ factors})$$
$$= a^m b^m$$

The remaining four are just as easy to demonstrate.

When these laws have been carefully demonstrated for positive integral exponents and are thoroughly understood, fractional, negative and zero exponents should be taken up. The treatment at this point should be as brief *as possible*. In case the pupil does but a single year's work in algebra, *it would be better to define the symbols* $a^{\frac{p}{q}}$, a^{-n}, a^0 *outright without any*

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(8) On this point the teacher should consult for his own satisfaction one or more of the following books: Chrystal, *Text-book of Algebra* Vol. I, pp. 25-29; Fine, *College Algebra* p. 57; Weber-Wellstein *Encyklopädie der Elementar Mathematik* Bd. I, p. 27. One might choose for the three laws I, III, IV, or II, III, IV, or I, III, V, or II, III, V. In any case the other two laws may be derived from the three and in that sense are not independent laws.

explanation whatever, then to inform him that the five laws already familiar apply to all exponents and assign a sufficient number of exercises to enable him to gain some facility in their use. Nor is this method open to serious objection for the prospective college student.

The worst possible method of procedure is to attempt to prove $a^{\frac{n}{r}}=\sqrt[r]{a^n}$, $a^{-n}=\frac{1}{a^n}$ and $a^o=1$ making no assumptions whatever. These forms receive their meanings *through definition.* At the outset such forms as $a^{\frac{n}{r}}$, a^{-n}, a^o are absolutely meaningless. since they cannot be interpreted in the light of the definition already given for a power. Too much emphasis can scarcely be put upon this point, for no progress can be made with the student who looks upon the power as a thing already completely defined. It does not help matters to say, for example, that

$$a^n \div a^n = a^{n-n} = a^o \text{ and also } a^n \div a^n = 1;$$

whence one must have $a^o=1$. This form of reasoning is incorrect, since it *assumes* that the result a^o *has* a meaning and this was deliberately excluded in the proof of the law.

$$a^m \div a^n = a^{m-n}$$

by subjecting m and n to the restriction $m>n$. Insist, then, that $a^{\frac{n}{r}}$, a^{-n}, a^o are meaningless when viewed in the light of the old definition, and cannot be used in any way until they are adequately defined.

To obtain adequate definitions one might proceed in the following manner: If the new forms are to be used as powers, it is necessary for us to make the new definitions generalizations of the old. This can be done by defining them so that they will obey the same laws. Now it has been shown that if we frame our definitions so that the new forms will obey one of the five laws, they will obey the other four. It is well then to make them obey the simplest, which would seem to be the first, or $a^m . a^n = a^{m+n}$. The assumption that the new form is amenable to this law is equivalent to a definition and we are no longer free, but must make the definition in accordance with what are now necessities in the case. We would have then, since

$$a^{\frac{1}{2}} . a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a,$$

$$a^{\frac{1}{2}} = \sqrt{a}$$

and similarly, since

$$a^{\frac{1}{3}} \,.\, a^{\frac{1}{3}} \,.\, a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=a,$$

$$a^{\frac{1}{3}}=\sqrt[3]{a}$$

and generally, since

$$a^{\frac{n}{r}} \,.\, a^{\frac{n}{r}} \cdots \text{—to } r \text{ factors} = a^{\frac{n}{r}+\frac{n}{r}} \cdots \text{—to } r \text{ terms} = a^n,$$

$$a^{\frac{n}{r}}= \sqrt[r]{a^n}.$$

The definition is, then, that the form $a^{\frac{n}{r}}$ denotes the rth root of the nth power of a. The other definitions will be obtained in a similar fashion.

It would have been just as effective to assume some other of the five laws, say $(a^m)^n=a^{mn}$ In this case we would have had in a particular case

$$(a^{\frac{1}{2}})^2=a^{\frac{2}{2}}=a,$$

whence we see again that $a^{\frac{1}{2}}$ must be defined as a quantity whose square is a; hence it must mean $\sqrt{a}$, with similar results in the general case.

It could now be shown that these new forms follow the remaining four laws. However, this is not advisable with high school pupils. It would be better by all means simply to say, and to emphasize the fact, that *all computations with exponents are effected by means of the five laws, and to set the class at work upon problems involving the use of the laws.* The translation of formulas expressing the laws into words that are clearly understood is especially important in the case of the laws for exponents. They should be read forward and backward until they are indelibly fixed in the minds of the pupils. For the pupil who would have some skill in handling algebraic expressions they are just as necessary as the multiplication table is for the ordinary computer.

When once the pupil knows the meaning of fractional exponents and has some skill in using them, he is ready for the solution of practically any problem that may arise in radicals, for the radical is nothing but a number which is otherwise expressed by means of a base affected by a fractional exponent. Future problems of this sort may be first translated into the language of exponents and then solved.

Ratio, Proportion, and Variation

Ratio and proportion have already received some attention in the arithmetic. They may now be taken up with some completeness, though nothing need be said at this stage concerning the ratio of incommensurable numbers. The fractional form will be much more convenient and easy to manage in computation than the older colon form. Indeed there is no good reason for making a sharp distinction between a ratio and a fraction so far as elementary work is concerned.

The number of necessary propositions is quite small. Given

$$\frac{a}{b}=\frac{c}{d}$$

one needs for ordinary work no more than the following:

1) $ad=bc$

2) $\dfrac{a}{b}=\dfrac{mc}{md}$

3) $\dfrac{b}{a}=\dfrac{d}{c}$

4) $\dfrac{a}{c}=\dfrac{b}{d}$

5) $\dfrac{a+b}{b}=\dfrac{c+d}{d}$

6) $\dfrac{a-b}{b}=\dfrac{c-d}{d}$

7) $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

Such propositions as that relating to the proportion between the squares of the terms will follow at once, since both sides of any equality can be squared, while the propositions relating to the proportionality of the roots of the terms will follow under the same limitations as must be imposed in extracting the root of both sides of an equality.

The teacher will find that the class will take little interest in the subject of ratio and proportion unless the work is made

as concrete as possible. To this end it is advisable to bring in a good number of illustrative examples from the affairs of everyday life. Some of the newer elementary algebras contain excellent lists. If these are not at hand, the teacher will find a large number of good problems in the arithmetics. It will do no harm to review these.

Variation is but another form of proportion. For example, if x_1 pounds of sugar cost y_1 cents and x_2 pounds at the same rate cost y_2 cents, we have at once

$$\frac{x_1}{x_2}=\frac{y_1}{y_2},$$

and by alternation of terms

$$\frac{y_1}{x_1}=\frac{y_2}{x_2}.$$

Similarly, for any other number of pounds x costing y cents, we shall have

$$\frac{y}{x}=\frac{y_1}{x_1}=\frac{y_2}{x_2}.$$

Whence it follows that the ratio of the number indicating the cost to the number indicating the quantity is constant. In symbols this statement is

$$\frac{y}{x}=c, \text{ or } y=cx.$$

The equation $y=cx$ may then be taken as a definition of direct variation and we say "y varies as x."

We might have taken the matter the other way around for if we define direct variation by $y=cx$, then from $y_1=cx_1$ and $y_2=cx_2$, we have at once

$$\frac{y_1}{y_2}=\frac{x_1}{x_2}.$$

Additional light will be thrown upon the subject if the matter be put in graphical form and the pupil be made to understand that the graph represents the table of variations of y dependent upon x. The pupil who knows that the graph of $ax+b$ is always a straight line will see very quickly that di-

rect variation is always represented by a straight line, since if y varies as x we have

$$y = cx$$

and cx is a particular case of $ax+b$.

There are many other ways in which one number y may depend upon another x; one of the most important of these is inverse variation, defined as follows; y varies inversely as x when it is equal to a constant times $\frac{1}{x}$ or

$$y = c\,\frac{1}{x}$$

For students who expect to take up physics or applied mathematics, the subject of variation should be taken up later for review and extension.

IV

THE THIRD HALF YEAR'S WORK IN ALGEBRA

The Review

The third half year's work in algebra should begin with a rapid review of factoring, with the quadratic expression chiefly in mind. This should be followed by a thorough review of quadratic equations with one unknown with emphasis on the more difficult problems and exercises on equations that are solved like quadratics. More work will be done with the solution of a type form such as

$$x^2+px+q=0$$

$$x^2+px+q=0$$

and the character of the roots will be determined from the solution

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

The connection between this form of solution and the equation in the form

$$(x-r_1)\ (x-r_2) = 0$$

will be pointed out.

The definition of the imaginary unit as the square root of a negative number, or a number whose square is negative

must be given and in this connection the significance of the expression $\frac{p^2}{4}-q$ and its relation to the character of the roots will be noted. The theorems regarding the sum and the product of the roots are useful and interesting though not absolutely essential.

Computation with Imaginary Numbers

The solution of so simple an equation as

$$x^2+1=0$$

brings in a new kind of number, namely, one whose square is negative. This number is fundamentally different from the ordinary positive and negative numbers whose squares are always positive. In as much as these numbers occur very frequently in the solution of quadratic equations, it is necessary that pupils have some knowledge of the methods used in computation with them.

It is best for the sake of brevity to use the notation

$$\sqrt{-1}=i$$

The fundamental steps are then two in number, as follows:

(1) Every number of the form $\sqrt{-a}$ can be reduced to the form ci where c is an ordinary rational or irrational number. In symbols we have $\sqrt{-a}=\sqrt{a}\,\sqrt{-1}=\sqrt{a}\,i$ in which case $c=\sqrt{a}$.

(2) $i^2=-1$.

If these two principles be thoroughly mastered and ordinary care exercised, the rest is easy. It is not necessary to give complicated examples. Products and quotients involving such expressions as

$$x+a+bi$$

will give all necessary drill. Products of the form

$$(x-2+3i)\ (x-2-3i)$$

are especially useful and the pupil should be taught to look at such expressions as the product of a sum and a difference and to give the result quickly as

$$(x-2)^2+9.$$

The imaginary can always be removed from the denominator

by a process similar to that employed in reducing fractions with irrational denominators. For example

$$\frac{1}{x+yi}=\frac{x-yi}{x^2+y^2}.$$

It is unwise to go into the geometrical interpretation of imaginaries except possibly with small classes of exceptionally strong pupils and even with such pupils care should be taken *not to go too far into this work at the expense of other things of greater importance.*

The Third Step in Graphical Representation.

In connection with the study of simple equations of the form $ax+b=0$, the graphs of the functions $ax+b$ were studied. As a part of the work in quadratic equations we may study the graph of the function

$$ax^2+bx+c$$

The study of this general form should be begun by first studying the graph of the function x^2; then ax^2+bx and finally ax^2+bx+c may be considered. In each case the pupil will form a table like that here given for x^2. The range of x is here given as from -4 to $+4$. This range may be increased at will for either positive or negative values. The pupil will have no difficulty in transferring the figures in the table to his coordinate paper to form the graph. He may be told the name of the curve, the parabola, but any study of its properties is for the present out of the question.

x	x^2
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

Some bright boy may be interested in the graph of x^3 and even of x^4. It will be interesting to him to compare a series of such graphs. In the study of the graph of ax^2+bx+c one of the most important things to be brought out is the relation between the roots of the equation $ax^2+bx+c=0$ and the points when the graph crosses the x-axis. Strong emphasis should be put upon the fact that the roots of the quadratic equation are identical with the two values of x at which the graph crosses the x-axis and attention should be called to the similar result in the case of the graph of $ax+b$. The geometrical

significance of equal roots and of imaginary roots is important also.

As special cases, one may find the graphs of such functions as $16t^2$ which gives the space through which a body falls from rest in the time t.

Quadratic Equations with Two Unknowns

A very brief review of simultaneous linear equations with the emphasis on elimination by the method of substitution will serve as an admirable introduction to simultaneous quadratics. Only the simpler cases can be taken up since in the general case the elimination of one unknown leads to an equation of the fourth degree in the other unknown. The first case to be taken up is the linear-quadratic system. Following this will come the case in which both unknowns enter in the second degree but no term of the form xy is present. A single other case, namely, that in which no terms of the first degree occur is about all that should be attempted.

For the last mentioned class, the solution is most easily accomplished by the process indicated in the following example: If we wish to solve the system

$$2x^2-3xy+5y^2=3$$
$$3x^2-7xy+y^2=4$$

we may multiply the first by 4 and the second by 3 then subtract, i. e., elimate the constant terms. The result is

$$x^2-9xy-23y^2=0.$$

This equation when divided by y^2 takes the form

$$\left(\frac{x}{y}\right)^2-9\frac{x}{y}-23=0$$

which is a quadratic equation in $\frac{x}{y}$. Solving this quadratic one obtains an equation of the form

$$x=(a\pm\sqrt{b})\,y$$

This last equation together with either one of the original equations forms a linear quadratic system which can be solved easily.

It may be said in passing that the subject of simultaneous quadratics is frequently over-emphasized in our American text-books. It is not of great practical importance either for

elementary or advanced work and the problems are too much on the puzzle order to be in keeping with the real spirit of algebra. The French program of study for boys does not even mention the subject. One of the best of the German hand books devotes three pages to the subject. The Weber-Wellstein *Encyklopädie der Elementar-Mathematik* devotes three and one-half pages to it but puts it *after* the treatment of equations of the third and fourth degree. A recent American book called a "first course in Algebra" gives forty pages to the same subject.

The Fourth Step in Graphical Representation

If the teacher feels that time will allow, the subject of simultaneous graphs may be taken up. Nothing will throw more light upon the real nature of simultaneous equations.

Hitherto graphs have been looked upon as *graphs of functions.* The transition to the *graph of the equation* is easy. The graph of the function $ax+b$ is a straight line. If we introduce the new variable y defined by the equation

$$y=ax+b,$$

we have an equation of such character that it is satisfied by the coordinates of every point on the line. The line which is the graph of the function $ax+b$ is then the graph of the equation

$$y=ax+b.$$

Any "linear" equation (and the meaning of the term "linear" is now clear) may be reduced to the form

$$y=ax+b$$

by transposition and division, unless, indeed, the coefficient of y should be zero to begin with, and its graph can be constructed. The special cases for which the coefficients of y and of x are zero must be considered separately. The graph may be constructed by means of a ruler when *any* two of its points are known. Usually the points that are easiest to find are the two intersections with the coordinate axes.

If then we have the graph of any equation

$$ax+by=c,$$

we can draw the graph of any other equation

$$a_1x+b_1y=c,$$

with the same coordinate axes. These two graphs will in gen-

eral intersect in a point whose coordinates give the solution of the system of *simultaneous* equations.

Simple as this matter is the inexperienced teacher will find that it will require considerable skill and no end of patience to get it clearly before a class. The idea ought to be presented again and again with a sheet of coordinate paper and a ruler in the hands of every member of the class. The graphical solution of a system may be looked upon as an end in itself but the added light that it throws upon the notion of a system of simultaneous equations is the most important part of it. With the graphical solution clearly in mind, it is possible to introduce the notion of incompatible equations, i. e., systems of the form

$$ax+by=c_1$$
$$ax+by=c_2$$

through the parallel graphs. The pupils can be made to see also that three equations linear in two variables are, generally speaking, not consistent because their graphs do not in general all pass through a single point.

The systems

$$ax+by=c$$
$$y=mx^2+nx+b$$

and

$$ax+by=c$$
$$x^2+y^2=m^2$$

may be taken up and studied in the same way. It will be noted that the graph of the equation $x^2+y^2=m^2$ is the same as the graph of the function $y=\pm\sqrt{m^2-x^2}$.
It is even possible to carry the matter further by considering the graphs of equations having the form

$$ax^2+by^2=c$$

and of systems of such equations. Such work is, however, scarcely profitable for high school pupils but belongs rather to the domain of analytical geometry.

Exponents and Logarithms

Students who have had a year of algebra should be able to demonstrate the five fundamental laws for integral exponents as given on p. 38 above, and should be able to make any ordinary computations with expressions involving negative and

fractional exponents. As a preliminary to the study of logarithms, the subject of exponents should be reviewed carefully and somewhat extended. A few of the fundamental laws may be proven for fractional and negative exponents although a clear statement of the laws is perhaps all that is necessary. For example, one can easily prove that

$$a^{\frac{m}{n}} \div a^{\frac{r}{s}} = a^{\frac{m}{n}-\frac{r}{s}}, \text{ when } \frac{m}{n} > \frac{r}{s},$$

or that

$$(a^{-m})^n = a^{-mn}.$$

It is not recommended that the teacher take up the whole series of propositions for the gain is not sufficient to compensate for the work involved. A somewhat fuller exposition of the theory of radicals may be given, though this is not wholly necessary.

Important as the technique of computation with exponents may be, by far the most important thing in this particular connection is the subject of logarithms. The first fact to be mastered by the pupil is that a logarithm is an exponent, no more, no less.

The easiest method of approach will be to construct a table of integral powers of 10 thus:

$$\begin{aligned}
10^{-3} &= .001\\
10^{-2} &= .01\\
10^{-1} &= .1\\
10^{0} &= 1\\
10^{1} &= 10\\
10^{2} &= 100\\
10^{3} &= 1000
\end{aligned}$$

every line of which has the form $10^x = y$. This table can be extended at will in either direction. From it follows by *definition*, which must be clearly given for the special case,

$$\begin{aligned}
\text{log. } .001 &= -3\\
\text{log. } .01 &= -2\\
\text{log. } .1 &= -1\\
\text{log. } 1 &= 0\\
\text{log. } 10 &= 1\\
\text{log. } 100 &= 2\\
\text{log. } 1000 &= 3
\end{aligned}$$

where it is understood that we are always considering powers

of 10. Similarly one may construct a table of powers of any other number, for example 2, and a corresponding set of logarithms. By means of such a table one may perform certain calculations with great ease by simply referring to the table. For example, one has $2^7=128$ and $2^{11}=2048$. To find the product of 128 by 2048 we have only to look in our table and find out what number corresponds to 2^{18} which has been computed in advance. Indeed we may drop the 2 from consideration and fix our attention upon the exponents 7, 11 and 18 and the numbers corresponding to them. Such examples will be only dimly comprehended at first, but keep them before the class for some time and come back to them after the general theory has been taken up.

To generalize the foregoing results, we assume that in the equation

$$a^x=y,$$

where a is a positive number different from unity, for a given value of x there is always one real positive value of y and conversely for a real positive value of y there is always one real positive value of x. In other words, to every logarithm corresponds a number and to every number corresponds a logarithm. The teacher may *not* frame this assumption in words but make it tacitly. There is no danger that it will not be accepted. The main point to insist upon is that the equation has exactly the same form as any line in the above table of powers of ten. Consequently to this equation corresponds another, namely,

$$log_a\, y=x,$$

which is to be found in the table of logarithms corresponding to the table of powers. Here we are always thinking about powers of a. This number a is called the *base* and is ordinarily taken to be 10. We have then the following definition of a logarithm: *A logarithm of a number is an exponent which indicates the power to which a given base must be raised to produce the given number.*

This much accomplished, it remains to place in sharp relief the following five propositions upon which the work will be based:

(1) The logarithm of the base is unity.

(2) The logarithm of unity is zero.

(3) The logarithm of a product is the sum of the logarithms of the factors.

(4) The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

(5) The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent whether the exponent be positive, negative or fractional.

These propositions are as easy to demonstrate with the general base a as with the special base 10.

From the very beginning the pupil should be taught to write all negative logarithms with —10 instead of a simple negative characteristic, as for example

$$\log.2 = 9.3010 - 10$$

rather than

$$\log.2 = \bar{1}.3010.$$

This form presents no difficulties except when it is necessary to divide the logarithms and none in this case if one takes the precaution to add and subtract such a multiple of 10 that the quotient of the negative part by the divisor is —10. For example,

$$\begin{aligned}(\log.2) \div 6 &= (9.3010 - 10) \div 6\\ &= (59.3010 - 60) \div 6\\ &= 9.8835 - 10\end{aligned}$$

The writer very much prefers to use the arithmetical complement when logarithms are used in division but the teacher of high school pupils will need to use some judgment in the matter, and if it appears too complicated for the class, omit it.

A very large number of practical problems are available for solution by logarithms and the wide-a-wake teacher will make the most of these. Mensuration, interest, both simple and compound, physics, mechanics, and many other subjects will furnish a list from which excellent problems can be taken.

Four-place tables are sufficient for all ordinary purposes and require much less time than five- or six-place tables. The linear arrangement will require much less time for explanation, and a table of anti-logarithms will be of great service. Tables printed on card-board are admirable since with a moderate sized page there is little or no turning of leaves.

Great care should be taken to secure neatness and accuracy

as well as rapidity and the pupils should be made to understand the limitations of logarithms. For example, by means of four-place logarithms, one may find the exact product of 28 by 35 but if one attempts to find the product of 28750 by 35975 by the same tables the result may be in error by as much as 5000.

The slide rule will be an important aid in exciting interest in the subject of logarithms. It consists essentially of a pair of scales exactly alike upon which the divisions are logarithmic but the numbering is arithmetic. By sliding one scale over the other, logarithms are added or subtracted as desired and the results can be read off at once. The accuracy of the rule obviously depends upon its length. A ten-inch rule will give results about as accurately as a three-place table of logarithms if used carefully. Such a rule would cost from $2.00 to $3.50 according to the quality, and it ought to be a part of the apparatus of every high school. If it is thought desirable, a large demonstrator's rule ten feet in length and with divisions large enough to be read by a class seated in an ordinary room could be obtained from slide rule manufacturers. The cost ought not to exceed $25.00.

It would not be advisable to require that all members of a class procure rules nor would it be advisable to spend a great amount of time in explaining the use of it but if a good rule is accessible to the boys and a little help is given, many of them will soon be able to make practical use of it and their knowledge of logarithms will certainly be advanced at the same time.

Ratio, Proportion, and Variation

Mention has already been made of this important group of subjects and it is to be presumed that the student has some knowledge of ratio and proportion. The review that is necessary will be in the line of preparation for problems in variation. At the out-set the pupil should be made to understand that variation is only another aspect of proportion. Indeed there is no very good reason for the introduction of the term variation since it is just as easy to say that one magnitude is proportional to a second as to say that the first varies as the second. To say that the interest of a given principal varies as the time means that the ratio of the interests for two given

times is equal to the ratios of the two times. Or if the space traversed by a moving body is proportional to the time one has

$$s : s_1 : : t : t_1$$

where s, and t, is any given pair of values and s and t any pair whatever. If we consider the measures [1] only we may write

$$\frac{s}{t} = \frac{s_1}{t_1}$$

from which it follows that

$$\frac{s}{t} = \text{a constant, say, } c,$$

or $$s = ct$$

or s varies as t, according to the definition of direct variation given above, p. 42. The graphical representation of such variation has been given.

In a large number of problems that are of fundamental importance in physics we have magnitudes whose variation depends upon functions of other magnitudes. The most important of these are:

(1) Variation as the square,

$$y = cx^2,$$

which is the law giving the distance through which a body falls from rest in the time x,

(2) Inverse variation,

$$y = c\frac{1}{x},$$

which is the law expressing the relation between the pressure and the volume of a gas at constant temperature.

(3) Variation as the inverse square,

$$y = c\frac{1}{x^2},$$

which is the Newtonian law of gravitation. If in general y varies as any function $f(x)$ of x, we should have

$$y = cf(x),$$

which equation furnishes a definition of variation in its most general form. One of the important things to be noted in this

[1] With beginners, it is not necessary to be too particular about the rules of the very formal people who always insist that only magnitudes of the same sort may be used as terms of a ratios. A little common sense and a good course in the dimensional equations of physics and mechanics will make the matter clear to the teacher.

connection is that the constant c is determined by a single pair of values of x and y. For example, if one seeks to determine the constant in the law for falling bodies and should find by experiment that in two seconds a body falls 64.4 ft. we have only to make $x=2$ and $y=64.4$ in the equation

$$y=cx^2$$

to find that c is 16.1. The determination of the constant in nearly every case depends upon data determined by experiment.

Besides these we may have the case where one number varies jointly as two or more variables or as any function of these. Such cases are, however, too complicated for presentation to the elementary students.

Another Step in Graphical Representation

So far, the graphs that have been constructed have been chiefly of functions of the types

$$ax+b \text{ and } ax^2+bx+c,$$

or what comes to the same thing graphs of equations of the type

$$y=ax+b \text{ and } y=ax^2+bx+c.$$

There is one other important class of graphs that should be constructed of which the graph of the gas law equation when the temperature is constant, is typical. The function which gives the pressure p at constant temperature when the value v is known is $\frac{k}{v}$. We have then to find the graph of the equation

$$p=\frac{k}{v}$$

This is easily done when k is given by constructing the table of values as has been done in former cases. This form is a particular case of the more general form

$$y=\frac{ax+b}{cx+d}$$

as may be seen by making $y=p$, $a=0$, $d=k$, $c=1$, $d=0$. It may easily be shown that for a given set of numerical values of a, b, c and d the new graph is similar to the graph of $\frac{k}{v}$,

the most conspicuous difference being one of position. These graphs differ from the ones that have already been constructed in that they have an "infinity" as well as a "zero" and these two important points can be determined from the function itself without constructing the graph. Namely, the "zero" is the root of $ax+b=0$ and the infinity is the root of $cx+d=0$. It will probably interest some of the brighter members of the class to find the graph of the Newtonian law.

The discussion of graphical representation here presented may seem to some teachers unnecessarily full. It has been given at some length for two reasons. The first is that it was desired that proper emphasis be placed upon the graph as the graph of a *function* rather than as the graph of an *equation*. It is just this "functional way of thinking" that Professor Klein has emphasized so strongly in Germany in recent years, and which has been given such an important place in the program of the French secondary schools. The second reason is that it seemed necessary to mark out rather more sharply the ground to be covered in this subject than in others since it is newer and teachers are less familiar with it. Several instances have come to the writer's notice where entirely too much time was devoted to graphical work.

The teacher should keep clearly in mind the two main reasons for the introduction of graphs, viz., the study of functions of a single variable, as $ax+b$, ax^2+bx+c, $\frac{ax+b}{cx+d}$, and the study of simultaneous systems of equations. The study of the functions is for the high school pupil much the more important of the two.

Finally, it is to be hoped that the work in graphs will be incorporated as a really essential part of the work and not made to appear as an appendix to be taken up if time allows. In its simplest form it should be put before the student at an early period and kept before him as one of the important aspects of the work.

The Binomial Theorem

The binomial theorem with positive integral exponent can be demonstrated by elementary students in two ways, by means of the theory of permutations and combinations or by

the method of complete induction. The latter method is the one which is most easily available for high school work and it has the additional advantage of being the only example that the elementary student is likely to get of one of the most important methods in mathematics.

In the first place we note that the expansion

$$(x+a)^4=x^4+4x^3a+6x^2a^2+4xa^3+a^4,$$

may be written in the form

$$(x+a)^4=x^4+\frac{4}{1}x^{4-1}+\frac{4(4-1)}{1.2}x^{4-2}a^2+\frac{4(4-1)(4-2)}{1.2.3}x^{4-3}a+$$

$$\frac{4(4-1)(4-2)(4-3)}{1.2.3.4}x^{4-4}a^4.$$

This would lead us to guess that when n is a positive integer we would have

$$(x+a)^n=x^n+\frac{n}{1}x^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3+\ldots.$$

But this last expression is nothing more than the result of a guess and, moreover, it is incomplete since there are other terms to be added.

Suppose, however, we multiply both sides of this identity by $x+a$. If we take care to reduce the result on the right side to its simplest form, we shall find that we have

$$(x+a)^{n+1}=x^{n+1}+\frac{(n+1)}{1}x^na+\frac{(n+1)(n)}{1.2}x^{n-1}a^2+$$

$$\frac{(n+1)n(n-1)}{1.2.3}x^{n-2}a^3+\ldots.$$

where the n of the previous form is replaced everywhere by $n+1$. From this it follows that if the form we have taken for the nth power is correct for a single value of n, it is correct for every greater value of n. We are then assured of the correctness of the form for it satisfies the requirements when $n=1$.

The demonstration is not yet complete since we have not

yet found the general term in the expansion for $(x+a)^n$ and we do not know what form it would take in the expansion for $(x+a)^{n+1}$. The teacher may omit this part of the demonstration if he thinks best. However, it would be a mistake to omit the determination of the general term.

When the theorem has been demonstrated for positive integral powers, the teacher may state that under certain limitations the same form of expansion holds for fractional and negative exponents as well as for positive integral exponents and require the pupils to carry out the expansion in a few such cases.

It will help the brighter pupils to realize the great generality of the theorem if they are led to discover for themselves that $(1-x)^{-1}$ will give the same result as the division $\frac{1}{1-x}$; that $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ can be used for extracting the square root of such numbers as .8, .9, 1.1, 1.2. .98, 1.03 and that similar things can be done in the way of extracting the cube root by the expansion for $(1+x)^{\frac{1}{3}}$.

Progressions

The subject of progressions has great intrinsic value and it is not difficult to teach. It furnishes an excellent review of simultaneous equations and moreover, it forms the best possible introduction to the theory of infinite series.

After the pupils are well grounded in the definitions, two theorems in arithmetical and two in geometrical progressions are needed. These theorems give the formulas for finding the nth term and the sum of n terms. All problems should be solved by considering these two formulas as simultaneous equations which may be solved for any two numbers when three of the five which enter into consideration are given.

The case of the geometrical progression in which the ratio is less than one and the number of terms is increased indefinitely will require some care. The teacher should be careful to give a definition of a limit that while it may not be complete, will not need to be unlearned later on. The necessary theorems concerning limits may be assumed without statement so that the pupils may go directly from the expressions for last term and sum with a finite number of

terms to the similar expressions when the number of terms has been increased without limit by saying that

$$\begin{aligned} \lim l &= \lim ar^{n-1} \\ &= a \lim r^{n-1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{and } \lim S &= \lim \frac{rl-a}{r-1} \\ &= \frac{r(\lim l-a)}{r-1} \\ &= \frac{a}{1-r}. \end{aligned}$$

A table of successive powers of some proper fraction, say $r=.5$ or $r=.1$ will help pupils to appreciate the fact that the expression r^{n-1} approaches zero when r is a proper fraction and n is increased indefinitely.

The application of geometrical progressions are numerous and of very great importance. The theory of rent, compound interest, annuities, and portions of the theory of life insurance, all are based on geometrical progression. The theory of repeating decimals finds its justifications here also. The teacher will have no difficulty in finding appropriate problems.

V.

ADVANCED ALGEBRA

In a few schools a course in advanced algebra is given covering a half year's work. Such a course should be preceded by trigonometry and may properly begin with a brief course in permutations and combinations. Topics may be selected from the following list:

1. The theory of determinants with applications to the solution of systems of simultaneous linear equations and the theory of elimination.
2. The theory of complex numbers including graphical representation.
3. Variables and limits (elementary treatment).
4. Permutations and combinations.
5. The elements of the theory of probabilities.

6. The theory of undetermined coefficients.

7. The theory of equations, including the factor theorem, the remainder theorem, relation between roots and coefficients, the determination of commensurable roots and Horner's method for the approximation to incommensurable numbers.

8. Binomial equations.

The subjects are given in about the order of their relative importance. Of course the treatment will in every case be quite elementary. It is unnecessary to take up the subjects in detail since most of these subjects have no place in the ordinary high school in its present state of development.

IV.

GEOMETRY

Introductory

It is scarcely possible to go into the course in geometry in such detail as has been given for the algebra work without reciting the list of propositions suitable for high school work. This does not seem to be necessary since a number of excellent lists have already been published and are available for the use of teachers.[1] These lists represent, for the most part, the reaction that has taken place in recent years against the great amount of work that has been put into the text-books in geometry. The object has been in every case to eliminate non-essentials and to lay proper emphasis on the important parts of the work. It is to be hoped that these reports have opened the way to a material simplification of the work in both plane and solid geometry as represented in many of our

[1] *Manual of the Free High Schools of Wisconsin*, Fifth Edition (Revised), Madison, 1906, pp. 44-53. Contains a report of a committee of the mathematics conference of the State Teachers' Association presented in December 1904.

Final report of the Committee on the Fundamental Propositions of Elementary Geometry presented to the Association of Mathematical Teachers of New England. Boston, 1906.

Preliminary Report of the Committee on Geometry. Presented to the Central Association of Science and Mathematics Teachers, 1906. G. W. Greenwood, Chairman. (Plane Geometry only.)

Geometry Report, Central Association of Science and Mathematics Teachers. School Science and Mathematics, January, 1909, pp. 73-78.

recent text-books. There is no question but that the situation would be greatly improved if a smaller amount of work were demanded and that work more thoroughly done.

The value of geometry as a subject for high school work is not seriously questioned at the present time. No part of mathematics yields results more valuable or more universally used, and none furnishes greater opportunity for exercising and testing the pupil's ingenuity and for giving him efficient drill in exactness of statement. No other science furnishes examples of such rigorous proofs based upon such simple premises and no subject except perhaps drawing will do more to strengthen the powers of visualization.

Mention has already been made of the value of some sort of concrete geometry as a subject of the grammar school curriculum. While this is being done more and more through the later improved arithmetics, just as in the case of the algebra, the geometric work done in the grades can never be taken for granted when the class is started in geometry. There is no danger that edge will be taken from the curiosity of the pupil by reason of the fact that he has already some acquaintance with the fundamental things in geometry. On the contrary, he will take up the matter with greater zest because it gives promise of going deeper into things that have already come to his attention. It is urged that whether the pupils have been trained in concrete geometry or not, the teacher should procure a copy of some of the excellent books upon the subject and work through it carefully as the class progresses in the regular text.

Probably the mistake that is most frequently made by the inexperienced teacher is in going too rapidly at the beginning of the work. Some teachers apparently divide the number of pages to be covered by the number of working days and then take just so many pages per day regardless of the character of the work. Nothing could be more depressing in its effect upon the slower pupils who must have time to get themselves accustomed to the geometric method. A class that needs a whole day for a single proposition in September may easily take three more difficult ones in the same time the following May.

Another error which, unfortunately, is sometimes made a

hobby by experienced teachers, consists in looking at plane geometry as a set of propositions forming an unbroken chain which is to lead by the shortest path to the determination of the numerical value of the ratio of the circumference of a circle to its diameter. In a similar fashion, for such people. the end to be attained in solid geometry is the measurement of the three "round bodies." Such a narrow conception of geometry is most unfortunate. The quadrature of the circle is only an incidental matter in the great domain of geometry and from most points of view is not a whit more important than the properties of the ellipse and parabola, many of which are equally attainable by elementary methods.

Definitions

At the outset the student of geometry is confronted by a very considerable body of definitions which he is supposed to read and digest and to possess as a part of his mental furniture ever afterward.

Certainly a few definitions are necessary to begin with, but the number is frequently made much larger than it need be and the pupils are required to take them all at a sitting whether the number be large or small. It would be better if the set of definitions were looked upon as a list to which the pupil could go as the definitions are needed for the work. It scarcely needs to be said that in any case the class will need to return to them again and again as his work progresses, until the necessary definitions are indelibly fixed in the mind of every pupil.

Some of the definitions frequently given are meaningless to the class and must remain so for a long time. It sounds very learned of course to begin with the definition of a definition, then proceed by successive steps to the definitions of mathematics, arithmetic, algebra and geometry, but such learned and philosophical attempts to "pigeon-hole" the universe make no appeal whatever to the boy or girl. They do harm by making the pupil spend his time on things he cannot understand. As has already been said, the teacher must take great care not to assume too much knowledge on the part of the pupil. This fault is one to which bright young teachers who have gone well into their subject are particularly liable. In-

deed it is one of the common faults of teachers from the kindergarten to the university.

If care is taken to assign work which will draw out the pupils and stimulate them to think for themselves, the explanations can scarcely be made too elementary. It does not follow that one must spend much time upon the things that the pupil has had in the grades, but if the pupil has forgotten how to reduce two arithmetical fractions to a common denominator, it is worse than useless to find fault with him or ridicule him for not knowing it. The thing he needs is help, not criticism.

The teacher must be warned that there are certain fundamental concepts in geometry that are defined with very great difficulty, if indeed they can be defined at all. In such cases it is hardly fair to expect the pupil to obtain a firm grasp of the concept by twenty minutes' study. The fundamental concepts of geometry are four, namely, the notion of a point, of a straight line, of a plane, and of an angle. It would be theoretically desirable to define these four things with perfect clearness at the outset and then build up a logical system upon these definitions and the axioms. But one has only to examine a half dozen texts on geometry to see what apparent confusion reigns in the definitions of these things. The difficulty is inherent in the subject. Every person who has reached high school age has a more or less accurate notion of a straight line and yet mathematicians from Euclid down to the present day have striven with indifferent success to put the definitions in such form that their accuracy could not be questioned. Euclid's own definition of a straight line, as usually interpreted, applies equally well to the helix, a curve which, roughly speaking, is shaped like a cork screw. (2) One can not define a straight line as that which divides a plane into two congruent parts as Leibnitz did, since this presupposes the definition of the plane. It can not be defined as the shortest distance between two points, as Archimedes defined it, for a line is not a distance. Moreover, the notion of distance is one of the most difficult to formulate of all the notions with which

(2) Enriques, *Encyklopädie der Mathematischen Wissenschaften*, article on *Prinzipien der Geometrie*, Bd. III, p. 18.

mathematics has to do. Neither does it seem wise to define a straight line as the shortest line between two points for the term line is itself difficult to define. Objections of various sorts may be brought against practically every definition of a straight line that has been given. In a similar way objections may be made to the current definitions of an angle, though perhaps not to the same extent.

Enough has been said to indicate the nature of the difficulties that present themselves in the matter of definitions. What is the teacher to do? Some teachers impressed with the magnitude of the difficulty have insisted that we must bring in certain "indefinables" i. e., notions for which the student has initially clearer conceptions than can be conveyed by attempted definitions, these conceptions being derived directly from experience. This method of procedure is advised by the Committee of the Central Association of Science and Mathematics Teachers whose report has already been cited. This committee gives as a partial list of such indefinables the following terms: "Point," "line," "surface," "plane," "straight," "whole," "part," "between," "in," "through," "divide." In his great papers on the foundations of geometry, Hilbert begins with an *Erklärung* (explanation) of which the first paragraph may be translated as follows: "We consider three systems of things: The things of the first system we call points and designate by A, B, C, — — —; the things of the second system we call lines and designate by a, b, c, — — —; the things of the third system we call planes and designate by α, β, γ — — . The points are also called the elements of linear geometry, the points and lines are called the elements of plane geometry, the points, lines and planes are called the elements of the geometry of space." No further attempt is made to define the terms point, line and plane.

It does not seem necessary to abandon all attempts because it is difficult to give an accurate definition. There is no serious objection to the use of many of the definitions in current use provided the pupils' notions are tested and corrected from time to time. A boy's knowledge may not be greatly advanced by being told that "a straight line is such that any part of it, however placed, lies wholly in any other part if its extremities lie in that part," but if he is told that when a weight suspend-

ed by a cord is allowed to come to rest (if such thing be possible) the cord becomes approximately a portion of a straight line; or that the edge of a cubical block of wood is somewhat like a line, he is apt to say that he knew as much before. If, however, he is told that the stretched string will not answer as an example of a straight line because it is thick and clumsy like a tree trunk for example, or that the edge of the cubical block will not answer because when one puts it under a magnifying glass he can see "kinks" or "crooks" in it, and moreover that even though the objections noted were removed, neither would do because they are not indefinitely extended in both directions, (3) he is probably making some little progress. It will be by just such illustrations, with constant correction that the boy's notion of the ideal geometrical line is made clear and definite, rather than by the elaborate attempted definition. Time is an essential element in the process.

It will not be understood of course that the same difficulty is to be found in the definitions of all the terms employed in geometry. If one assumes that the terms line and point are known, the expression "intersection of two lines" is defined with perfect clearness and accuracy. The same would be true of a triangle or a circle and indeed of most of the terms employed.

The difficulties mentioned above have not been pointed out with a view of discouraging the teacher or of encouraging lax and slovenly methods. It is believed that when the teacher knows what the difficulties are he will be incited to greater effort, that he will see more clearly that even in the beginnings of geometry, knowledge does not come in a flash, and that he will understand that geometry is not quite such a cut-and-dried piece of work as some of our cock-sure writers of text-books would have us believe.

There is another class of definitions in geometry which, though they are entirely clear, ought to be revised for the sake of the student who may perchance carry his mathematics beyond the high school. One of these is the definition of the

(3) It has been assumed throughout that the line and the plane are indefinite in extent. Although we habitually use, and will probably continue to use, such expressions as the "line AB" rather than line segment or line piece, mathematicians are nearly unanimous in favor of the use of the term to indicate a figure indefinite in extent.

word "equals." The word is used in two senses. In one case, two figures are said to be equal when they are equal in area and in the other they are said to be equal when they can be made to coincide throughout. It would make for clearness if the word equal when used in connection with surfaces and solids should always be accompanied by a modifying phrase, as "equal in area" or "equal in volume," and if the term "congruent" were to be used for figures that can be made to coincide throughout. Not only does the indiscriminate use of the word "equals" lead to confusion of thought but it may even lead to statements which are untrue. For example, suppose one has two material congruent triangles and from each of these triangular pieces which are congruent are cut. The figures that remain are equal in area but they can not be made to coincide throughout unless the triangular pieces are cut from both in exactly the same manner. The axiom which asserts that "if equals be subtracted from equals the remainders are equal," is true in this case only if we understand the word "equal" to mean "equal in area." The confusion is somewhat lessened if one uses the expression "equal in every respect," or "identically equal." There is no reason, however, why the term congruent should not be used for figures that may be made to coincide in all points. The term is brief and to the point and is sanctioned by use in many other parts of mathematics.

In elementary geometry the circle is almost universally defined as "a portion of a plane bounded by a line every part of which is equally distant from a point within it," but if we turn to any text in analytic geometry we will find that the circle is defined as "the locus of all points equally distant from a given point." This will probably be true even though the same author has written both books. Exactly similar things may be said of the definition of a sphere. To put the matter otherwise, the terms circle and sphere mean a portion of a plane and a portion of space respectively in elementary geometry while they mean two wholly different things in all branches of mathematics which the boy may take up later on. Some of our friends who may have had trouble with certain things in graphs and wonder why the boy has difficulty in understanding that the equation

$$x^2+y^2=r^2$$

is the equation of a circle, have probably never stopped to consider that no equation exists which is satisfied by just the points of the geometrical figure that the boy has in mind and by no others.

The same criticism applied to the definitions of many other terms employed in geometry. A polygon is defined as "a portion of a plane bounded by" etc., a polyhedron as a portion of space, and so on to the end of the list. It would seem to be better to define polygon as a closed figure made up of "line segments" or "sects" lying in a plane and that the polyhedron is a closed surface made up of plane surfaces. The cone and the cylinder would be surfaces and not solids. (4)

Two additions to our list of terms to be defined which are recommended by the Committee of the Central Association of Science and Mathematics Teachers are "ray" to mean one of the two parts (5) into which a line is divided by one of its points, and "sect" to mean the finite portion of a line which is determined by two points. These two recommendations have much to commend them. The term ray has already been in use for many years in projective geometry and its use makes possible a very simple and useful definition of an angle, viz., an angle is a geometrical figure composed of two rays extending from a point. The other has the merit that it defines exactly the thing the boy is likely to think of as a line. It will help him to clear thinking if he is required to define a "sect" or "line segment" or "line piece" and then is made aware of the fact that a line is something else.

Finally, there are three fundamental concepts whose definitions come under no one of the foregoing classes which are likely to give the teacher some trouble. These concepts are the concepts of length, area and volume.

For the beginner, it is probably sufficient to say that the measure of the length of a line segment is the number of times it contains the unit of length, but the teacher, at least should

(4) The writer is acquainted with but a single American book in which the author carries out the changes here recommended. This book, Holgate's *Elementary Geometry, Plane and Solid*, published in 1901 by the Macmillan Company, is one of the most scholarly books that has appeared recently. The Committee of the Central Association of Science and Mathematics Teachers recommend the change for the circle but carry the matter no further.

(5) Strictly speaking a line is not divided into two parts by one of its points, since a point might move from one side of the given point to the other by passing through the infinite point.

know that this definition fails whenever the unit of length is not contained in the given length. Strictly speaking, the definition would be meaningless in so simple a case as that in which the length to be measured is half a unit to say nothing of the case where it is incommensurable with the unit.

If the length to be measured is a portion of a straight line, one may say "the measure of the length is the ratio of the given line segment to a line segment of unit length." This definition, however, presupposes an intimate knowledge of incommensurable ratios, which subject is treated elsewhere.

If we are dealing with portions of curved lines, such as an arc of a circle, the above definition fails utterly. If the line were a flexible, material thing which could be straightened out without stretching, the difficulty would be overcome, but geometric curved lines cannot be so manipulated and it is a rather violent assumption concerning the properties of space to suppose that we can do so. Accurate thinking upon this point demands the formulation of a fairly accurate definition of length which shall be applicable to curved lines, for this concept cannot be classed among the "indefinables." The definition which has been agreed upon by mathematians is, when applied to an arc of the circle, substantially as follows: "The length of an arc of the circle is the limit toward which the perimeter of a convex broken line inscribed in or circumscribed about a circle and terminated by the extremities of the arc, tends, when all the sides are diminished indefinitely." (6) The situation is exactly similar with regard to the definitions of area and of volume and the difficulty would be surmounted in a similar way.

It must be admitted that these notions have in them some intrinsic difficulty but the clear-headed, painstaking teacher will be able to do much to bring order out of the confusion of thought that is in the pupil's mind. It can not be done all at once. In geometry, as well as in other things, it is the injunction of the prophet, "precept upon precept, precept upon precept; line upon line, line upon line; here a little and there a little" that will accomplish the desired results. Absolutely clear thinking on the part of the teacher is the first requisite

(6) Hadamard, *Leçons de Géométrie Élémentaire* I p. 173. Holgate and MacMahon, among American writers, give similar definitions, but for the most part the text-book makers ignore the question entirely.

to success; patience and common sense in the matter of knowing how far the pupils are able to understand the matter, will accomplish the rest.

Axioms and Postulates

Two things stand out clearly as a result of recent discussions on the teaching of geometry. One is that it is advisable to admit a considerably larger body of axioms than is absolutely necessary to furnish a secure logical basis for the subject. If, for example, the pupil thinks (with Euclid) that the proposition that all straight angles are equal or that all right angles are equal does not need proof but is self-evident, let him accept it as true and go on to some thing the truth of which does not appeal to him so clearly. Similarly, it may be sometimes expedient to use a proposition which has not been proven. This is certainly good pedagogy if the teacher takes care to see that the content of the new proposition is clearly understood. In the French program of secondary instruction one frequently finds such expressions as, "the fact will be given without the theory," "the value of π will be announced simply."

The other thing is that the axioms and postulates should be introduced as needed and not all in a body. The books from Euclid down to the present time have followed this plan with the definitions but the pupil has had to take the axioms and postulates in a single dose. The result has been that they have been learned in a perfunctory way at the beginning and forgotten until the reference or the "hint" says "axiom" when the number is given glibly enough but no realization of its meaning. The mere printing of the axioms and postulates at intervals throughout the book will not by any means meet the difficulty. They must be taken up and discussed when they are needed, regardless of the position they may happen to have in the text. It would be an excellent plan to have them printed in a body and besides at various points in the book.

There is at the present time a strong tendency among teachers to use the single term "assumption" to include both axioms and postulates. While there is no particular advantage in the introduction of a new term, there is no good reason why the distinction between axiom and postulate should be

maintained. The postulates occupy exactly the same relation with respect to problems as axioms do to theorems.[7] This is especially true since it is now known with some certainty that some propositions have been transferred from one class to the other since Euclid's time. The word assumption is well suited to replace these terms since it conveys exactly the meaning that is desired.

The teacher who is aware of the real difficulty that attends the determination of a set of axioms for geometry will not be too particular in trying to use the minimum number with elementary pupils. In recent years many of the most eminent mathematicians of the world have worked upon the problem of finding a complete system of independent axioms. In 1899, Professor Hilbert of Göttingen published a profound paper entitled *"Grundlagen der Geometrie" (Foundations of Geometry).* In this paper he lays down five groups of axioms, twenty in all. In 1902, Professor Moore proved that one of these axioms was superfluous in that it could be proven from the others. In 1904, Professor Veblem presented a set of twelve axioms. In 1902, Professor Hilbert published a second paper in which he uses three axioms the content of which is unintelligible to anyone who has not gone a good way into advanced mathematics. Pasch, Poincare, Veronese, Lie, and many others have published important papers upon the same subject. These sets of axioms vary greatly. For example, no one of the three axioms laid down in Hilbert's second paper is among the twenty given in the first paper. Again, several of Hilbert's axioms *can be proven as theorems* on the basis of the Veblen axioms, and conversely. Euclid more than once assumes propositions not found among the twelve axioms, sometimes tacitly and sometimes openly.

What does all this have to do with the teaching of high school geometry? the teacher is apt to ask. Simply this: The axioms of geometry are not restricted to twelve propositions, no more and no less, definite and fixed in content, but may vary widely in number and in content. The discussion of sets of axioms is a matter that is far beyond the reach of the ordinary high school teacher. The teacher is concerned chiefly with a

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[7] Casey, *The Elements of Euclid.* 5th Ed. 1887.

set of axioms that contain no contradictions. Whether they be too few in number or too many, it will be impossible for him to decide. *The wise teacher will then use as many as he sees fit, usually a number considerably larger than the lists given in the text-books, and applying the name axiom to only such propositions as appeal to the boy as being incontrovertibly true in the world with which he is acquainted.* It may not be necessary in all cases to formulate the axioms. [8]

It must not be understood by any means that the foundations of geometry are not an appropriate subject for study by the teacher. Exactly the reverse is true. The sensible teacher will be greatly stimulated and his work materially strengthened by wide and careful reading in this direction. Such reading will not tend to over-refinement or extra rigor of treatment of geometry, as Professor Young wisely remarks, but it will free him from the bondage of the text-book by making him see that what he has before him is certainly incomplete and it may not be the best by any means.

Some of the Things to be Emphasized in Geometry

The triangle, upon whose properties nearly all stable construction, whether in architecture or practical mechanics, is based, should receive careful attention. The propositions relating to the congruence of triangles and those later propositions which relate to the similarity of triangles are especially important. The former, as ordinarily presented, furnish an easy introduction to that most important method known as the method of superposition, while the latter lead up to the most important propositions in mensuration. Indeed, the criteria for the similarity of two triangles are in some respects the most important things in elementary geometry. The theorems of the first group and the theorems on parallelograms easily deduced from them are fairly easy and form an admirable introduction to the whole subject.

The properties of the circle are of fundamental importance in the later work in geometry and in nearly all of the higher mathematics, since it is by means of the circle and the ruler that all, or nearly all of our geometrical constructions are

[8] Young, *The Teaching of Mathematics.* Longmans, Green & Company, New York and London. 1907. p. 200.

made. The constant π is one of the two most important occurring in mathematics. Emphasis should be laid upon the relations of angles inscribed in a circle. It is unfortunate that our text-books in geometry leave the matter of measurement of angles almost entirely on one side. The discussion of the relation of the angle to the subtended arc is in most cases made fuller than need be while nothing is said of circular measure which plays such an important part in later mathematics. Indeed the subject of measurement receives much less attention in our mathematical work than it deserves.

The subject of ratio and proportion is one of the most important and at the same time one of the most difficult in geometry. The difficulties arise from two sources: The first source of difficulty lies in the fact that the theory of proportion as ordinarily given is wholly arithmetical while the pupil has to do with geometrical applications, and proper care is not taken to bridge the gap between the arithmetical and the geometrical theories. The second source of difficulty lies in the fact that the pupil for the first time comes into close quarters with incommensurable numbers. It is not advisable, even if it were possible, to lead a class through all the intricacies of incommensurable numbers, but the treatment ought to be as rigorous as the circumstances will permit. A special section is given entirely to this subject. (See below pp. 75-82.) The importance of the idea of a ratio lies in the fact that it lies at the basis of all work in measurement of geometric magnitudes. It is the real basis not only for all that is practical in geometry but for much of the theoretical part.

Closely connected with the subjects of ratio and proportion is the subject of similar figures. Many of the propositions relating to similar figures are intrinsically valuable but the method is even more important. The student will get more of the real spirit of geometry from this part of the work than any other, and it is the inculcation of this spirit which will go farther toward making the work valuable and enjoyable than anything else.

It is scarcely necessary to call attention to the fundamental character of the proposition which asserts that the areas of any two rectangles are to each other as the products of the measures of their bases and altitudes. The propositions relat-

ing to areas of plane figures of whatever shape are practically corollaries to this theorem.

The Pythagorean proposition and the set of propositions which may be grouped about it need special mention although they are developed by means of the work already mentioned, under the head of areas of polygons. The teacher who is interested in this historic proposition will find abundant material for his own study and for the stronger members of the class who may have time to do some extra work. Several scores of demonstration have been discovered for this proposition, most of them elementary but a number of them depending upon various parts of higher mathematics.

In this connection, it would be profitable for the teacher to point out the fact that the two propositions which are usually given, relating to the squares on the sides opposite an acute or an obtuse angle of a triangle respectively are generalizations of the Pythagorean theorem and reduce to it when the acute angle or the obtuse angle reduce to a right angle.[9]

The measurement of the circle is very important but the most important thing from the standpoint of pure geometry under this head is not the solution of the problem to find the approximate value of π as many teachers suppose, but the theorem which asserts that the ratio of the circumference to its diameter is constant. The exercise that the students get from the solution of this famous problem is excellent and the uses to which the result is put are very numerous, but the mechanic will not remember it any better for having gone through the long computation, and the advanced student of mathematics will have half a dozen opportunities to obtain the same result with less expenditure of labor. It ought to be noted clearly, however, that the object of this book is the measurement of the circle and that the numerous propositions on regular polygons are merely subordinate to the main purpose.

[9] "This booke moreover contayneth two wonderfull propositions, one of an obtuse angled triangle, the other of an acute; which with the ayde of the 47+ proposition of the first booke of Euclid, which is of a rectangle triangle, of how great force and profite they are in matters of astronomy, they knowe which have trauayled in that arte. VVherefore if this booke had none other profite beside, only for these 2+ propositions sake it were diligently to be embraced and studied." *The Elements of Geometry of the most auncient Philosopher Euclide of Megara Faithfully* (now first) *translated into the Englishe toung by H. Billingsley Citizen of London.* (1570). Translator's introduction to the second book.

A limited number of geometrical constructions will be of very great value. It is worth while to require the pupils to put their constructions upon first class paper (regular draughting paper is best) and to insist that they shall be made as neat as they can be made with ink and a fairly good ruler and dividers. Pupils should be taught to test their ruler by drawing a line and then applying the ruler to the line in reversed position.

It is important that the teacher should insist on the essential simplicity of elementary geometrical constructions. All constructions possible in elementary geometry must be accomplished by means of two fundamental ones, viz., the construction of a straight line determined by two given points and the drawing of a circle with a given radius and a given point for its center. The most important constructions are, the erection of a perpendicular to a given line, the duplication of a given angle, the construction of a fourth proportional, the construction of a mean proportional and the construction of a square whose area shall be equal to the sum or the difference of the areas of two given squares. These open the way to practically all the others.

There is no good reason why a scale, a triangle, and a protractor, and even a T-square might not be added to the instruments used in constructions provided care is taken to point out the significance of the use of these instruments. It would be of very great advantage to the work in geometry if it could be done in connection with some simple mechanical drawing. All pupils should be taught to draw to scale either in connection with the geometrical constructions or the regular drawing work.

Some Applications of Geometry

Teachers of geometry will, of course, be familiar with the ordinary applications of geometry to mensuration and will make free use of them as opportunities present themselves. There are many useful things besides problems in mensuration that will add much to the interest in the work.

In connection with the work in similar triangles, a most useful exercise is the determination of the height of a tree or a building by means of two perpendiculars of known length

having their upper extremities in line with the point whose height is to be measured, and the distance between their feet. To solve this problem, one has only to arrange the perpendiculars so that their upper extremities are in line with the point whose altitude above the plane is sought and to take the proper linear measurements. If the object whose altitude is sought is surrounded by a comparatively level surface, the problem can be solved by means of a measuring tape and a plumbline for setting the perpendiculars.

The above problem brings the pupil into contact with some of the fundamental problems of the surveyor, viz., the problem of determining a perpendicular to the horizontal plane, the alignment of three objects in a plane, the accurate measurement of a length. Pupils may be required to make several determinations of the problem and to make a profile drawing drawn to scale. If the drawing is made on squared paper, the pupils will be surprised and delighted to find that no computation is necessary to obtain an approximate result.

Another useful problem, also involving similar triangles, is the problem of making an exact copy of a map. The map should be simple, giving the location of not more than half a dozen objects and perhaps a stream course. The pupils should be required to note the difference between the two problems, making a copy the same size, and making a copy differing in size and also to realize clearly that in the latter case they really have a problem of similar triangles. After a map has been copied, it will be a simple step to make a map of a given piece of ground that is irregular in shape and to locate objects upon it. These problems can all be solved by means of a scale, a pair of dividers and a measuring tape. A steel tape not less than fifty feet long should be used if it can be obtained.

The teacher will find some useful information upon this topic in most elementary form in Mair's chapters, "To make a copy of a map" and "On drawing to scale" in his volume, *Practical Mathematics*. (See bibliography.) Hadamard's *Leçons de Géométrie Élémentaire* also contains three short chapters upon this subject that give many useful suggestions.

Another interesting problem is the approximate determination of the latitude of a point on the earth's surface. It can be solved by the use of a perpendicular of known length, a

measuring tape, a carpenter's level and a protractor to measure the angle when a profile drawing has been made to scale. This problem will materially assist the boy to get a firmer notion of parallel lines.

This list of problems available for the purposes of elementary instruction can be added to almost indefinitely. It is not recommended that much time be taken from the book work for such problems but a few of them will do much to arouse interest.

Limits and Incommensurables

By common consent the most difficult propositions in geometry are those which involve incommensurables. Teachers have sought for some method by which the use of limits could be avoided in the demonstration of such propositions and it has been proposed to abolish the notion of a limit from secondary mathematics.

The difficulty is a very real, but by no means an insurmountable one. That it is impossible to avoid the use of the concept of a limit without changing very radically the character of the instruction in geometry, is seen when one notes that without it one can not demonstrate rigorously a single mensuration formula. The length of a curve, the area of a surface, either plane or curved, the volume of a solid—all are in the last analysis *defined* as limits. It would seem, then, to be impossible to abolish the concept of a limit if we are to continue to teach geometry as a rational subject.

This being the case, the method of procedure is clear enough. In the first place, the teacher must be thoroughly prepared by reading and study extended far beyond the textbook in hand, to meet every difficulty that may present itself in the work. In the second place, the work itself must be simplified as far as is possible without sacrificing wholly the rigorous character of the demonstration. Even after this has been done it will require all the skill the inexperienced teacher can command to present the subject in such a way that the ordinary pupil will be able to understand it. [10]

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[10] It is out of the question to go into the *theory* of limits with a class of high school pupils; such work is only suitable for mature students of college grade, but fortunately geometry requires practically no more than the definition of a limit and one or two theorems concerning limits. These elementary notions occur in

The notion of a limit presents itself in two very different ways. The first way, if one follows the ordinary text-book, is

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such simple connections that propositions involving them may be mastered by any high school pupil of ordinary ability and perseverance under the guidance of a teacher who has some skill and a fair amount of patience.

To begin with, it is necessary that the teacher should have clearly in mind the relation that exists between limits and incommensurable numbers. All the propositions involving limits are propositions involving measurement. The measure of a quantity is a number which has a definite relation to a number system. This number system, which is primarily algebraic, serves as a medium of expression for most of the facts concerning magnitudes whether they are geometrical or not. By reason of the correspondence between the numbers of a system and magnitudes, i. e., the distances of the points of a line from a given point, which correspondence is assumed to be exact, or in more technical language, "one-to-one," we may use the number or the measure and the magnitude interchangeably. The theory of limits, of which such a small portion is used in elementary geometry, is almost purely arithmetical in character. Even the geometric theory of exhaustions which was used with such power by Archimedes is not wholly free from the terminology of arithmetic. It follows from what has been said that the teacher of geometry needs to have some knowledge of the number system of arithmetic and algebra. The first five sections of Fine's *College Algebra*, will serve as an excellent introduction to this theory. The classical paper of Dedekind is accessible in an excellent English translation by Professor Beeman under the title "*Essays on the Theory of Numbers*," published by the Open Court Company.

We may note briefly that the arithmetic of positive numbers in its broadest aspect is concerned with, first, positive integers; second, rational numbers, or ordinary fractions; third, irrational numbers as $\sqrt{2}$, π, etc. The rational numbers are defined as quotients of two integers. The irrational numbers cannot be expressed in terms of integers or rational numbers but must be *defined* as the limits of sequences of rational numbers. For example, the symbol $\sqrt{2}$ is wholly inadequate when considered as the symbol of a number. It stands rather as the symbol of an operation which can not be completely performed, and whose end result can never be reached. One definition is that $\sqrt{2}$ is the limit toward which the numbers of the sequence

$$1, 1.4, 1.41, 1.414 \ldots.$$

are approaching where successive numbers are obtained by the process of root extraction. It might be just as well defined as the limit of the sequence

$$2, 1.5, 1.42, 1.415 \ldots.$$

Similarly, the number π may be defined as the limit of the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159 \ldots.$$

The ratio $\dfrac{\pi}{\sqrt{2}}$ is defined by the sequence

$$\frac{3}{1}, \frac{3.1}{1.4}, \frac{3.14}{1.41}, \frac{3.141}{1.414} \ldots.$$

The root of the whole difficulty lies in just this fact, namely, that the incommensurable number can not be defined without some resort to a limiting process or its equivalent.

It is scarcely necessary to caution the teacher against bringing too much of the theory of incommensurable numbers before a class of high school pupils. But the teacher ought to know something of the theory.

found in the set of propositions which involve the so-called "incommensurable cases," while the second is met when one seeks to find the length of a circumference or the area enclosed by it. It is in the second form that the limit is most frequently met with in solid geometry. The notion of a limit is much more easily grasped in the second class of cases than in the first. For this reason it would certainly be good pedagogy to omit the incommensurable cases in the first-named class of propositions at least until the pupil has acquired a firm grasp of the notion by studying it in the simpler form. Indeed many excellent teachers are now advocating the complete omission of the incommensurable cases. With this view the writer concurrs though he believes that it would be a serious mistake to omit them from the text. By all means let pupils be made aware of the difficulties that exist at this point.

Whether one omits the incommensurable cases or not, the first step in preparation for the propositions that involve the notion of a limit is to put before the class as clearly as possible the notion of a variable number or variable magnitude. To this end a number of easy examples are necessary.

If one attempts to express $\frac{1}{3}$ as a decimal the result is a set of approximations

$$.3,\ .33,\ .333,\ .3333\ldots.$$

A set of numbers, like the above, which are formed according to some definite law and in which the fixed numbers follow one another in a definite order is called a *sequence*. Suppose now that we consider a number x, which is not fixed like the numbers of the sequence, but which can be identified with the successive numbers of a sequence. This number x is called a variable. A variable may then be defined as *a number, or a magnitude, which may be identified with the successive numbers of a sequence or set of numbers or magnitudes which are determined according to some given law.* We sometimes say the variable "runs through" the numbers of the sequence.

Suppose, for example, one takes a rod two feet long and capable of indefinite division. If we bisect it, then bisect the remainder, and so on, we obtain a set of pieces the sums of whose lengths are measured by the numbers

$$1,\ 1\frac{1}{2},\ 1\frac{3}{4},\ 1\frac{7}{8},\ 1\frac{15}{16},\ldots$$

respectively.

These numbers form a sequence. If we represent the measure of the sum of the pieces cut off by l, l is a variable which runs through the numbers of the sequence. Analogous statements will apply to the lengths themselves as well as to their measures.

It will be noted that, as the process goes on, l is always increased. Similarly, one may consider the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \text{ or } \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \ldots$$

which indicate the lengths remaining after the successive bisections. If r denote the measure of the length remaining, r runs through the numbers of this new sequence. Moreover, r always decreases. Moreover, the peculiar character of the first sequence

$$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, 1\frac{15}{16} \ldots$$

should be placed in the boldest relief. The differences between the numbers of this sequence and the number 2 (which, indeed, are exactly the numbers of the second sequence), diminish constantly and if we go far enough this difference can be made smaller than any number that can be assigned, and it will remain so if we go further.

The definition of a limit is but a concise statement in general terms of the foregoing. A definition which, though scarcely complete, is correct as far as it goes and is adequate for the purpose of elementary geometry is as follows:

The limit of a variable is a constant toward which the variable continually approaches as it runs through a sequence of values and in such way that the difference between the constant and the variable may become and remain less than any number or magnitude that can be assigned.

The notion is placed in much sharper relief when we think of the limit as the limit of a sequence.

In the example given above, 2 is the limit of the variable l or of the sequence

$$1, 1\frac{1}{2}, 1\frac{3}{4} \ldots$$

through which l runs.

The following additional examples will help to fasten the notion in the minds of the pupils:

The limit of the sequence

$$.3,\ .33,\ .333\ldots.$$

is one-third. Indeed the successive differences between $\frac{1}{3}$ and the numbers of the sequence are

$$\frac{1}{30},\ \frac{1}{300},\ \frac{1}{3000}\ldots$$

and it is evident that, whatever small number one may select, he can go far enough to find a difference smaller than the number selected. It will help the class very much if the teacher will ask members to give some small numbers and then show that in every case a difference that is smaller can be found.

Another important but much more difficult example is the limit of the sequence,

$$1,\ 1.4,\ 1.41,\ 1{,}414\ldots.$$

where the successive numbers of the sequence are obtained by the process for the approximate extraction of the square root of 2. It is easy to show that the successive numbers of the sequence always increase; that the square of every one is less than 2; and that the differences between 2 and the successive squares become smaller and smaller. It is not so easy to show that a limit exists as is actually the case. The limit of this sequence is *by definition* the square root of 2. This definition will be utilized later on.

It may be noted that, in every example that has been given the variable does not reach its limit. This is not necessarily true in all cases, though most of the text-books would have us believe that it is.

The fundamental theorems on limits are the theorems that are concerned with the criteria for the existence of a limit. One of the most important of these is the theorem that asserts that "if a variable never decreases and never exceeds a fixed number M, the variable approaches a limit which is either M *or some number less than* M." This theorem, which will be referred to subsequently as theorem A, applies in the case of a very large number of geometrical problems. For example, we see intuitively that the area of a regular polygon inscribed in a circle is less than the area of the square circumscribing the circle and that the area of the inscribed polygon never decreases as the number of sides increases. It follows at once

from the theorem that the areas of successive polygons approach a limit which is, of course, less than the area of the circumscribing square.

Another theorem which in most text-books is made the basis of the work in limits is the well known theorem that if two variables are always equal and each approaches a limit, their limits are equal. This theorem will be referred to as theorem B.

Theorem A *should be assumed without statement* since the tacit assumption of the theorem involves no error. Theorem B may be stated and its truth assumed, i. e., it may be treated as an axiom.

Some of the theorems in which the notion of a limit is used may be demonstrated in a fairly simple and satisfactory manner without using the concept at all. For example, in the excellent German text of Henrici and Treutlein, the theorem that "in the same circle or in equal circles equal angles are subtended by equal arcs" is demonstrated as follows: "If by the rotation of the plane the radius MA describes the angle AMA., then a second radius MB will describe an equal angle BMB, and after the rotation the angle AMB coincides with the arc AB." This very simple demonstration, which is accompanied by an appropriate figure, is made to depend upon two simple propositions on rotations, both of which are practically taken as self-evident. [11]

The theorem that asserts that the area of a rectangle is equal to the product of its length by its breadth is demonstrated substantially as follows: In the first place the theorem is proven for the case where the lengths of the sides are integral multiples of the unit of length, by noting the number of squares in each column and each row as we were taught to do in the arithmetic. Then "if the unit of length is not contained an integral number of times (I translate freely), one takes for the measure the *n*th part of the unit of length; the square corresponding to this measure is then the $\frac{1}{nn}$ part of

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[11] The book in question is well worth careful study by our American teachers. In the space of 584 pages, the authors have given not only the essential parts of plane and solid geometry, but they have made a good beginning in projective geometry, and have given a fair treatment of plane trigonometry, and of the simpler parts of analytic geometry.

the surface unit since it takes nn of the small squares to equal the surface unit. If then the base of the rectangle $a = \alpha \cdot \frac{1}{n}$ and the altitude $b = \beta \cdot \frac{1}{n}$ there are α such small squares resting on the base and there are β such strips then $\alpha\ \beta$ of the small squares: the superficial content is then

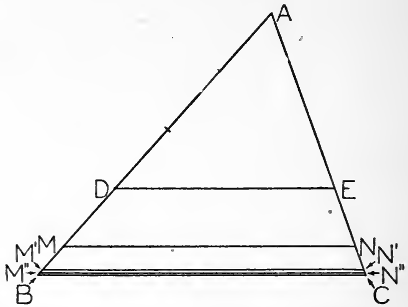
$$\beta \cdot \alpha \cdot \frac{1}{nn} = (\beta \cdot \frac{1}{n})(\alpha \cdot \frac{1}{n}) = ab$$

as in the first case. The fraction $\frac{1}{n}$ can always be taken so small that by the measurement of the sides at most an unmeasurable residue remains, which is not to be taken into consideration."

The above proof translated almost literally from one of the best German books is characteristic of German texts. To be sure it is not rigorous but the study of mathematics does not appear to have suffered appreciably in Germany by reason of the fact that German boys have been taught after this fashion.

The method employed in the foregoing proof may be used in all of the incommensurable cases. However, the teacher in the American school is ordinarily not at liberty to choose a book and, except for the most skillful teacher, it is a very questionable proceeding to substitute a proof not found in the text, especially if the new proof involves any particular difficulty.

If the demonstration of the incommensurable cases are taken up as they are given in most American text-books, the teacher's problem will be to get the pupil to realize the exact nature of the variable with which he has to deal. A numerical example will help much in this matter. Suppose, for example, that one has to demonstrate the theorem that asserts that "A straight line parallel to one side of a triangle divides the other two sides in the same ratio." Suppose the theorem has been demonstrated for the commensurable case. Suppose further that we have before us a particular case where the side AB is divided into two segments AD and DB which have the lengths 3 and $\sqrt{2}$. (One may tacitly assume the existence



of a line segment whose length is $\sqrt{2}$). The commensurable case applies to the triangle AMN and we have $\frac{AD}{DM} = \frac{AE}{EN} = \frac{3}{1}$ If now we take for the unit a length equal to one-tenth of the original unit, the ratio $\frac{AD}{DM}$ takes the form $\frac{30}{10}$ but is unchanged. But with this unit, it is possible to carry the measurement of AB nearer to B, say M′. The process of root extraction will show that we can measure 14 tenths along AB. With the new and smaller unit the commensurable case applies to the triangle AM′N′. We have $\frac{AD}{AM'} = \frac{AE}{EN'} = \frac{30}{14}$. Similarly we should find $\frac{AD}{DM''} = \frac{AE}{EN''} = \frac{300}{141}$. We have then the *two* variable ratios which take successively the values

$$\frac{AD}{DM}, \quad \frac{AD}{DM'}, \quad \frac{AD}{DM''} \cdots\cdots$$

and

$$\frac{AE}{EN}, \quad \frac{AE}{EN'}, \quad \frac{AE}{EN''} \cdots\cdots$$

respectively and any ratio of the one set is always equal to the corresponding ratio of the other set. The values which the variable ratio takes are $\frac{3}{1}, \frac{30}{14}, \frac{300}{141}, \frac{3000}{1414} \ldots$ where the unit of measurement is successively 1, .1, .01, .001. But the limit of this sequence of values is *by definition* the incommensurable ratio which is to be denoted by $\frac{3}{\sqrt{2}}$.

Such an illustration can not, of course, take the place of

a proof but it may help the pupil to understand the demonstration given in the books. [12]

Ordinarily more time is put upon the approximate computation of π than the subject warrants. The method used can be applied to nothing else and the methods of the infinitesmal calculus are incomparably more rapid. Most boys and girls would be quite as well off if the value of π were given to them and the time spent in getting an approximation to the sixth decimal place were put on something else. The fundamentally important fact is that the ratio of the circumference to the diameter is constant.

The computation of π should be abridged as much as possible. If the circumference of a circle be *defined*, as it should be, as the limit toward which the regular inscribed (or the regular circumscribed) polygon approaches as the number of sides is increased indefinitely, there is no need of proving that the circumference lies between the perimeters of the inscribed and the circumscribed polygon as is done in many texts. The simplest process depends upon the theorem which gives the length of the apothem and a side of a regular inscribed polygon having double the number of sides of a given regular inscribed polygon whose side is assumed to be known. If one has defined the area and length of a circle as the limits toward which the area and perimeter respectively of a regular polygon approach as the number of sides is increased indefinitely, the rest is easy. Most of the text-books give the proof of the formula

$$P_{2n}=\sqrt{\frac{d^2}{2}-\frac{d}{2}\sqrt{d^2-p_n^2}}$$

where p_n is the side of a regular inscribed polygon of n sides, d the diameter of the circle, P_{2n} the side of the regular inscribed polygon of $2n$ sides. By computing p_n in terms of d, one finds by the ordinary process the series of approximate values for the ratio $2n\frac{P_{2n}}{d}$. This sequence of numbers has for its limit the number π for which we have been seeking to find an approximation sufficiently exact for practical purposes.

[12] Teacher and pupil alike would be interested in Euclid's demonstration of this theorem. (Book VI, Prop. II.) The old Greek geometer avoided the explicit use of limits though the notion is implied in his use of the area of the triangle which depends upon the theorem for the area of the rectangle. Euclid did not go beyond the commensurable case in getting the area of the rectangle.

Assuming the fundamental theorem that a variable which never decreases and which is less than a given fixed number has a limit, the degree of accuracy of the approximation is easily seen.

Solid Geometry

The method of solid geometry differs little from that of plane geometry. If geometrical method were the sole end in view there is little to be gained by taking up the study of solid geometry after a thorough drill in plane geometry. There are, indeed, many teachers who believe that little additional benefit is to be gained from the study of solid geometry. With this opinion the writer can not agree. There are many persons who, initially, seem incapable of grasping space relations. For example, the three lines used to represent the three axes in an analytic geometry of three dimensions, mean to them three lines in a plane and nothing more. Such persons are almost wholly lacking in space-intuition. Perspective can have little or no meaning to them. Solid geometry, of all the studies to be found in the secondary school curriculum, mechanical drawing alone excepted, is best adapted to cultivate this space intuition. If the choice were to be made between these two subjects the wise teacher would not hesitate to choose the solid geometry as in every way more fundamental and more valuable.

Solid geometry is an absolutely essential part of the training of the engineer and for the worker in many lines of science. It is valuable for the mechanic for the light it throws upon the problems he meets in his daily work and for the mensuration formulas it places at his disposal, and to the general student for its vigorous discipline in sustained thinking and close discrimination.

Technical schools must always demand solid geometry for entrance and while there may be some question as to the advisability of requiring it of every pupil who completes a high school course, its intrinsic value ought to commend it to all who wish to be liberally educated.

The selection of the material of solid geometry in the textbooks in current use is, for the most part, excellent. The most important things are contained in the chapters usually devoted

to lines, planes and space angles, to polyhedra, and to cylinders and cones. These chapters are valuable because they contain the fundamental things of the subject, because they yield valuable practical results, and because they give the excellent examples of close reasoning without complicated demonstrations. The chapter on cylinders and cones is valuable as furnishing many examples in which algebra may be utilized. Possibly too much emphasis is put upon the geometry of the sphere by most of our texts. Some fundamental theorems on the sections of cones would be of more importance to the student who goes further with his mathematics and these are not much more difficult of demonstration than many of the theorems relating to the sphere.

The Use of Models in Solid Geometry

The use of models in the teaching of solid geometry is to be emphasized strongly. The models needed are inexpensive and many of them can be made by any boy who is handy with tools. They should be placed where the class can have free access to them and the pupils should be encouraged to handle them freely not only in the class room but also while they are preparing their lessons. Care should be taken, however, to see that the pupils are able to get the relations existing between the model and the book figure. From the standpoint of the engineer and the scientist, it is just as important that the pupil should be able to portray a solid figure on a flat surface as it is that he should be able in his imagination to make the figure pictured in the plane stand out in space.

A very simple modeling frame designed by Professor C. E. Comstock is described by Mr. E. R. Breslich in "*School Mathematics*" Volume 1, Number 1, p. 176. This frame which can be constructed by any teacher with ordinary mechanical ability, can be used for a great variety of purposes. In consists of a board two feet square, a number of round sticks one quarter of an inch thick and varying in length from six inches to two feet, several balls of twine of different colors and a few sheets of thin but strong card-board. The board is laid off in squares not less than an inch on each side and holes of diameter smaller than that of the sticks are bored at the corners of each

square. One end of the sticks may be notched to make it easier to attach the strings.

A very large number of the geometric solids may be built up by means of this frame. For example, to construct a pyramid one has only to insert a stick in one of the holes; then select a number of holes which lie at the vertices of the polygon that is taken as the base and pass strings through the holes and over the top of the stick. The pyramid may be either right or oblique and a very considerable variation in the form of the base is possible.

VII

TRIGONOMETRY

Trigonometry is admirably adapted to the later years of the high school course. The processes are simple and direct and the subject admits of many applications that will interest the pupils. It is easier than some parts of the algebra and has the additional advantage that it furnishes opportunity for drill in computation that is lacking in the algebra and geometry.

A good course in trigonometry should continue throughout a half year and should cover the following subjects:

1. A thorough drill in definitions of the trigonometric functions using the ratio and not the line definitions. The definitions should be so framed as to admit of easy and natural extensions to functions of angles that are greater than 90°.

2. A study of the relations that exist between the functions of an angle including the relations between the squares.

3. The determination of the numerical values of the functions of 0°, 30°, 45°, 60°, 90°.

4. The solution of right triangles with application to useful and interesting problems.

5. The generalization of the functions to angles of any magnitudes, positive or negative.

6. The study of the Addition Theorem and its consequences. Under the latter head one should take up functions of the double angle, functions of the half angle, the sum and the difference of two sines and the sum and the difference of two cosines. Functions of the triple angle may be taken up as exercises if desired.

7. The formulas used in the solution of oblique triangles.

8. The use of trigonometric and logarithmic tables.

9. The solution of oblique triangles, including the finding of areas of triangles.

10. A brief treatment of inverse trigonometric functions.

Each student should be required to work out a number of trigonometric identities for in no other way can the fundamental formulas be so throughly impressed upon their minds. In working out these identities it is not enough to work from both sides and thus simply verify the formulas. Rather, the exercise should be stated in the form of a problem. For example, in the books the pupil is asked to prove that

$$\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b=\cos^2 b-\cos^2 a.$$

A much better plan would be to require them to express $\sin(a+b)\ \sin(a-b)$ in terms of sines or of cosines.

The notion of expressing a trigonometric expression in terms of certain specified functions will be of much service to the pupil. Of course there are useful examples where the problem can not be stated in this form.

If the teacher so desires the formulas for the solution of triangles may be developed as needed rather than to group them altogether.

In the use of the tables care should be taken to get results as accurate as the tables will permit and to make clear to the pupils the fact that the degree of accuracy required depends upon the character of the data given.

Four-place tables will answer all requirements. It is advisable, however, to have a set of five-place or even six-place tables at hand for reference.

VIII

MISCELLANEOUS TOPICS

Some Suggestions for the Class Room

Although this bulletin is not intended to be a treatise on the teaching of mathematics, some suggestions regarding the actual class room management will not be out of place. No hard and fast rules for managing a class can be laid down but the teacher's action at every point must be governed by whatever judgment and experience seem to dictate. The teacher

who attempts to teach by rule soon gets into ruts. The following suggestions must then be taken as suggestions to be modified or disregarded altogether at the teacher's discretion.

1. There is, properly speaking, no problem in class-room discipline apart from teaching. In nine cases out of ten the trouble arises from the fact that the teacher does not know how to present the subject in such a way as to hold the attention of the class or the physical conditions are bad. A dull teacher, a poorly ventilated room, a noisy radiator, a blackboard so placed that it can not be seen are invariable sources of disorder.

2. In no other subject is care concerning the seating of the class and the arrangements of the furniture greater than in mathematics. As a rule, the class should be seated in a compact body and in such a way that every pupil can see the whole of the blackboard without turning around in his seat. The teacher should stand or sit squarely in front of the class except when pupils are explaining blackboard work.

3. The pupil who explains blackboard work should stand at the board and with pointer in hand should indicate every step as he proceeds with the explanation.

4. The explanation of the blackboard work should be regarded as a drill in oral presentation as well as an exercise in mathematics.

5. Ordinarily the recitation will begin with a brief review of the preceding day's work. This review may consist in a series of short, direct questions so framed as to bring out the important things in the work done or the teacher may sketch it clearly in a few sharply defined, clean cut sentences emphasizing leading points.

6. Frequent five or ten minute "written quizzes" consisting of a single question, or at most two questions, on the work of the days immediately preceding, will do much to keep the class keyed up to the point of efficient work. The questions should be fairly easy and completely representative of the work covered. The quiz need not be announced beforehand but it must be given in such a way that every member of the class will feel that he is fairly dealt with.

7. Longer written reviews should be given several times in a semester. The questions should be free from "catches" or unusual difficulties of any sort.

8. The greatest possible care should be taken to secure independent work from every pupil. It does little good to assign problems to a class if they are always worked out by a few of the stronger members and the solutions passed around to all the members of the class. Moreover, the line of demarcation between a slavish reliance on others and downright dishonesty is extremely difficult to draw. Dishonesty is the very last thing to be countenanced in school work. The judicious assignment of a few individual problems will help the teacher to correct the evil, or to prevent it entirely.

9. That indispensable piece of apparatus, the blackboard, is frequently abused. It was invented for the purpose of making it possible for a whole class to see a given piece of work at the same time, and not for individual work. At best, it is dirty and unsanitary. Where rapid drill in easy problems is desired, it is much better to have pupils work with pencil and paper seated quietly and comfortably at their desks, rather than have them work at the blackboard until everything in the room is covered with chalk dust and every pupil's back is tired. Results can be given just as well from the papers as from the black board and the teacher can see the work by passing quietly from desk to desk.

10. When the teacher uses the black board, the matter that is written must be read clearly as it is written. It is an excellent plan to encourage the pupils to "chalk and talk" at the same time, but care will have to be exercised as the ability to do this is acquired very slowly by some pupils.

11. Individual help outside class hours, while it is the most expensive kind of teaching, is the most effective and it will pay better in mathematics than in almost any other subject.

The Teacher

To the teacher who is obliged to work for six hours each day in the classroom, and to spend a considerable portion of each evening correcting papers, the problem of keeping up one's freshness and enthusiasm for the work is a most difficult one. But it must be met and solved somehow if the work is to be kept up to the highest efficiency.

The one absolutely essential thing is the maintainance of good health. Strict attention to one's daily regimen, to hours

of work and rest is quite necessary. One can not lose half the night's rest and guide a class through difficult propositions in geometry in a spirited manner next day. But similar observations would apply to any other subject as well as to mathematics.

It would probably be going too far to say that teachers of elementary mathematics are more likely to fall into ruts than teachers of other subjects. There can be no question, however, that mathematics, with its rather limited range of facts, which have to be presented in nearly the same fashion and to the same sort of pupils year after year, does not offer the same stimulus to the teacher and the same incentive to further study as do some other subjects. All the greater is then the need for corrective measures.

One of the important things for the teacher of any subject is a careful study of the special pedagogy of that subject.

The accessions to the literature on the teaching of mathematics have been unusually large during the last decade and the teacher who wishes to "read up" in this subject will find ample material. Young's *Teaching of Mathematics in the Secondary and in the Elementary School* will form an excellent basis for such study, since it contains very full bibliographical references up to the date of its publication in 1907. The columns of *School Science and Mathematics*, the *School Review* and other pedagogical journals will give references to the most important things among the current publications.

But it is the study of the subject itself that will do most for the teacher. It is often said that the teacher who knows a small amount of mathematics beyond the needs of the class room will do better than one who has gone a great way into the subject. The statement is on its face absurd, but if it should seem to be true in a particular case it is because the poorly prepared teacher must be a diligent student of his subject.

There is a considerable body of literature which is available for the teacher who wishes to look further into the subject of mathematics. Some of the outlines of work best suited to the high school teacher are here indicated briefly.

The history of mathematics is an admirable subject for private study by the teacher; there is now a goodly list of books

on this subject that possess a high degree of interest and the study of these cannot fail to be helpful. One of the simplest and at the same time one of the most readable books is Frankland's *Story of Euclid.* The histories of Ball and Cajori serve as admirable introductions to the general history of mathematics. Gow's *Short History of Greek Mathematics* will furnish material for the study of the development of Greek mathematics.

To most teachers of mathematics even though they have taken the regular courses in a first class college or university, it will be a revelation to learn that it is possible to have a system of geometry in which the sum of the angles of a triangle is not necessarily equal to two right angles. The study of Non-Euclidean geometry which deals with such systems should be of great interest and is not difficult. The original paper of Lobatchewsky translated by Halsted under the title, *The Science Absolute of Space,* is not beyond the reach of the average teacher. Professor H. P. Manning's *Non-Euclidean Geometry* will give further information, while the story of the development and significance of Non-Euclidean geometry is well brought out by Frankland in his *Story of Euclid.*

Comparable in a way to the study of Non-Euclidean geometry, is the study of algebras in which the product ab is not equal to ba. Kelland and Tait's *Quaternions* and Hyde's *Directional Calculus* will serve as easy introductions to these algebras.

For the teacher who has never had a course in projective geometry, the study of at least the simpler aspects of the modern pure geometry will be of interest. No better introduction to this subject can be found than the second appendix to Chauvenet's *Treatise on Elementary Geometry.* Lachlan's *Pure Geometry* will make a good second book and after this may come one of the two masterly treatises: Reye's *Geometry of Position* translated by Holgate or Cremona's *Elements of Projective Geometry* translated by Leudesdorf.

The theory of equations, a subject which is not always gone over in a college course, is valuable to the teacher by reason of the fact that the subject matter lies very close to several of the important parts of the high school algebra. Burnside and Panton's *Theory of Equations* is the best book for the student

who is working alone. The first volume of this book is especially valuable.

There is no subject, not immediately available for the purposes of high school instruction, which will be of greater service to the teacher than the theory of irrational numbers, for with this theory well in hand there is no excuse for confused thinking when the teacher comes to the incommensurable cases in geometry, to the theory of exponents, or to the relation of numbers and measurement. The best available introduction is probably the first seventy pages of Fine's *College Algebra.* The classical paper of Dedekind on *Continuity and Irrational Numbers*, forming a part of the volume entitled, *Essays on the Theory of Numbers* translated from the German by Beman and Smith, can be read with profit after one has read Fine.

It ought, perhaps, to be said that the teacher who is not tolerably well advanced in mathematics is likely to find a good many difficulties in his outside reading and it will be greatly to his advantage to be able to call upon someone who can help him over the hard places. Any member of the department of mathematics of the university will be pleased to lay out courses of reading or to explain any difficulties that may arise in the reading. It is to be hoped that a considerable number of our teachers will take up the matter of advanced reading in mathematics.

But more important than all these, is that kind of study by which the teacher seeks to enlarge and enrich his knowledge of the particular subject he teaches. To illustrate what is meant, let us consider for a moment the manner of presenting certain subjects in mathematics to high school pupils. In presenting the fundamental laws and the fundamental operations, irrational numbers, the theory of exponents, the application of limits in geometry, not to mention other subjects, it is only the barest outline that can be utilized for high school classes. It is, of course, possible for the teacher whose knowledge does not go beyond the text-book to go through the work in a mechanical fashion and even to make some show of efficiency in his teaching but such teaching can not possibly have the zest, the vigor, the enthusiasm, that comes from the work of the teacher who is full to over-flowing of his subject, whose constant effort is to hold back information rather than to empty himself of all he knows. It is the teacher who, from

the fullness of his knowledge, decides each day what to present and what not to present, that is able to inspire his pupils. And besides, how much more satisfaction will the teacher who knows his subject intimately and thoroughly get out of his work.

An adequate notion of our number system, for example, can not be presented to high school pupils. But the teacher who understands clearly that we begin with a unit, that we get a series of positive integers by the simple operation of counting; that we *invent* a fractional number so that division may be possible or so that certain magnitudes may be measured by means of a given unit; that we *invent* negative numbers because there are certain magnitudes which exist in two senses, or because we want the operation of subtraction to be possible in all cases; that after these new numbers are invented their sums, differences, products, and quotients, *must be defined anew in every case;* that after all these things have been done the number system is still inadequate because there are simple algebraic processes like the solution of the equation. $x^2-2=0$, whose results have no meaning, or magnitudes, like the diagonal of a square with unit sides that can not be measured; that the new numbers that must yet be invented will unlock most of the difficulties that arise in connection with limits in geometry,—the teacher who understands these things clearly, I say, will appreciate the limitations under which he works when dealing with high school pupils and will be in a position to deal intelligently with the outline presented in the text-book.

Such studies as have been indicated in the preceding paragraphs can not be carried on by means of the complimentary copies with which the text-book publishers may supply you. It will be necessary at the very outset for the teacher who wishes to make progress, to procure at least a few books that, in the matter of content are distinctly in advance of the ordinary commercial texts, books of the type of Fine's *College Algebra* or Chrystal's *Text-book of Algebra*, Gibson's *Elementary Treatise on Graphs*, Casey's *Sequel to Euclid* and others that might be mentioned. This point can scarcely be emphasized too strongly. The physician or the lawyer who does not

own a professional library is not accorded a standing either by his colleagues or by the community. The teacher ought to be willing to be judged by the same standard.

IX

BIBLIOGRAPHY

It has been difficult in years past to prepare a list of mathematical books suitable for high school libraries, owing to the small number of books that would be at once different from the ordinary commercial texts, and sufficiently elementary for use either by the teacher or the pupil. However, the last two decades have seen a very great increase in the number of available books. This is particularly true of books under the general head of practical mathematics and those dealing with the most elementary parts of geometry. The subjoined list contains a relatively large number of books belonging to these two classes for the reason that these two subjects are of great pedagogical interest at the present time.

All the books listed under the head of pedagogy and as many of the reports as possible ought to be in the private library of every person who proposes to make a serious business of teaching mathematics. *School Science and Mathematics*, which contains more material suited to the needs of the teacher of mathematics than any other journal published in this country, should come regularly to the teacher's table.

I have ventured to include in the list the title of a single book published in French. This book, Hadamard's *Leçons de Géométrie*, was written by an eminent French mathematician and is an illustration of what can be done in the way of getting pupils forward by judicious omission of non-essentials. It is one of the very best books that has ever been written on the subject of elementary geometry. It is a matter of great regret that it did not seem feasible to list such books as, Henrici and Treutlein's *Elementar Geometrie*, Weber and Wellstein's *Encyklopädie der Elementar Mathematik*, Tannery's *Notions de Mathématiques*, Tannery's *Leçons d'Arithmétique* and others which like these, have a very prominent place in the literature of elementary mathematics.

For the convenience of teachers who may wish to make a

selection from the list, the titles of fifteen of the more important books have been marked with an asterisk.

The most satisfactory way to order books dealing with highly specialized subjects is to write to one of the large book sellers indicating what is wanted and asking for prices.

Algebra and Theory of Equations

Burnside W. and Panton, A., *Theory of Equations.* Vol. 1. Longmans, Green and Company, New York. $1.90. A standard treatise on that part of algebra which lies just beyond the high school work.

Chrystal, G., *Algebra: An elementary Text-book.* Third edition; two volumes. Adam and Charles Black, London and Edinburgh. 1893. pp. 559 and 588. $7.25. One of the best algebras we have. Much of it is quite advanced but the first volume contains an unusually clear exposition of the fundamental laws and processes.

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*Fine, Henry B., *The Number System of Algebra.* D. C. Heath and Company, Chicago. 1891. pp. 131. $1.00. A clear account of the Cantor "sequence" theory of irrational numbers.

Hanus, Paul, *An Elementary Treatise on Determinants.* Ginn and Company, 1886. pp. 217. $1.80. An admirable introduction to this most important subject.

Woods, F. S. and Bailey, F. H., *A Course in Mathematics.* Vol. I. Ginn and Company. pp. 385. $2.25. One of the best books to show the teacher what lies ahead.

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*Failor, I. N., *Inventional Geometry.* The Century Company, New York. 1904. pp. 83. $0.60.

Course of Study in Geometry. A syllabus adopted by the Board of Superintendents of New York City. Department of Education of the City of New York. 1903. A course of study for seventh and eighth grade pupils.

Godfrey and Siddons, *Elementary Geometry.* Cambridge University Press. 1903. pp. 350. $0.87.

Hadamard, *Leçons de Géométrie Élémentaire.* Gauthier-Villars, Paris. 16 Fr. One of the best books of its kind ever published in any language.

Hall, H. S., and Stevens, F. H., *Lessons in Experimental and Practical Geometry.* The Macmillan Company, London and New York. 1905. pp. 94. $0.40. A most excellent little book, showing what may be done to introduce pupils to geometry.

Hanus, Paul, *Geometry in the Grammar School.* D. C. Heath and Company. 1904. pp. 52. $0.25. An excellent discussion of this subject.

Kerr, John G., *Constructive Geometry.* Blackie and Son, Limited, London. 1904. pp. 121. An excellent introduction to geometry covering practically the same ground that is covered by the first three books of Euclid.

Klein, Felix, *Famous Problems in Elementary Geometry.* Translated by Beman and Smith. Ginn and Company. 1897. pp. 80. $0.50. A brief account of the classical problems that have had most influence on the development of modern mathematical thought. Not easy reading.

Lachlan, R., *Elementary Treatise on Modern Pure Geometry.* Macmillan and Company, London. 1893. 9 sh.

*Myers, G. W., *Geometrical Exercises for Algebraic Solution.* The University of Chicago Press. 1907. pp. 71. $0.85. Eight hundred exercises designed to bridge the gap between algebra and geometry.

Row, T. Sundara, *Geometric Exercises in Paper Folding.* The Open Court Publishing Company, Chicago. pp. 148. $1.00.

Simson, Robert, *The Elements of Euclid.* This is the best English edition and can be had through any dealer in second-hand books. The editions published prior to 1830 are best.

Spencer, William George, *Inventional Geometry with a Prefatory Note by Herbert Spencer.* D. Appleton and Company. 1889. pp. 97. $0.35. This book was the forerunner of the numerous texts employing the heuristic method.

*Warren, A. T., *Experimental and Theoretical Course in Geometry.* Clarendon Press, Oxford. 1903. pp. 248. 2 sh. One of the best books of its kind covering the most important parts of plane geometry.

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Gibson, George A., *An Elementary Treatise on Graphs.* The Macmillan Company. 1904. pp. 183. $1.00. A very good account of one of the live subjects.

*Schultze, Arthur, *Graphic Algebra.* The Macmillan Company. 1908. pp. 93. $0.80. An excellent elementary presentation of this important subject, in which rapid methods for the solution of equations of the second, third, and fourth degrees are given.

Trigonometry, Numerical Computation, and Mathematical Tables

*Halsey, F. A., *The Use of the Slide Rule.* D. Van Nostrand and Company. 1903. pp. 84. $0.75.

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Lock, J. B., *Trigonometry for Beginners.* Revised by John A. Miller. The Macmillan Company. pp. 147. $1.00.

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which excludes the use of imaginaries may be obtained separately.

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Castle, Frank, *Workshop Mathematics.* The Macmillan Company, London and New York. pp. 329. $0.65. An excellent compendium of formulas and methods for the practical mechanic.

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Mair, David, *A School Course of Mathematics.* The Clarendon Press, Oxford. 1907. pp. 375. 3 sh. 6 d. The writer makes all school mathematics to consist of a series of problems.

Myers, G. W., *First Year Mathematics.* The University of Chicago Press. Third edition, 1909. pp. 181. $1.00. Includes topics in both algebra and geometry for first year pupils.

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*Ball, W. W. R., *History of Mathematics.* Third edition. The Macmillan Company, New York. pp. 464. $3.25. An accurate and readable book. One of the best of its kind.

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Cajori, Florian, *A History of Mathematics.* The Macmillan Company. 1907. pp. 304. $3.50.

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*Dedekind, Richard, *Essays on the Theory of Numbers.* Translated by W. W. Beeman. The Open Court Publishing Company, Chicago. 1901. pp. 115. $0.75. The essay on "Continuity and Irrational Numbers" is a classical paper and its contents should be known to every teacher of algebra and geometry.

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Poincaré, H., *Science and Hypothesis.* Charles Scribner's Sons, New York. 1907. pp. 244. $1.50. An extraordinarily lucid account of the foundations of mathematics and physics by one of the foremost of the modern mathematicians.

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Ball, W. W. R., *Mathematical Recreations and Essays.* The Macmillan Company, London and New York. Fourth edition. 1905. pp. 388. $2.25. Contains a large number of curious and interesting facts connected with mathematics.

Schofield, A. T., *Another World, or the Fourth Dimension.* Swan, Sonnenschein and Company. 1890. pp. 92.

Hinton, C. Howard, *The Fourth Dimension.* Swan, Sonnenschein and Company. 1906. pp. 270.

X

APPARATUS FOR THE MATHEMATICAL RECITATION-ROOM

A small amount of apparatus is indispensable in the mathematical recitation room. A larger amount can be used to good advantage if the teacher is properly qualified. The pieces that seem to me to be indispensable are given in a minimum list below. Prices from reliable catalogs are attached in most cases.

The item of first importance in every mathematical recitation room is, of course, the black board. The room should contain not less than thirty linear feet of black board and as much more as can be advantageously placed. One section near the teacher's desk should be ruled as described on p. 24. One eraser should be provided for every two or three pupils and every class entering the room should find the black board clean and the room as free from floating dust as possible.

Minimum List

Besides the black board the room should be equipped with the following articles:

Two rubber tipped pointers. 25c each.

Four straight edges, plain, 3 to 4 feet long. 50c to 75c each.

Yard-stick graduated to eighth inches. 50c to 75c.

Meter-stick graduated to millimeters. 50c to 75c.

Brass protractor (5 in). 75c.

Angle-meter (Becker). $1.60.

Assorted geometrical solids made of hard wood. $2.00.

Pair of crayon compasses with one rubber tipped foot. $1.00 to $2.00.

Box of crayons, assorted colors. $1.10.

Supply of squared paper, twenty divisions to the inch. 10c per quire.

Slated sphere 15 to 20 inches in diameter. $4.00 to $15.00.

For $8.00 to $15.00 one can buy a "full meridian mounted" sphere.

The lower priced spheres are mounted to rotate about one diameter only. This item would not be in a minimum list for schools that do not teach the geometry of the sphere.

Supplementary List

Slide rule, 10-inch. $2.50 to $3.50.

Drawing table, T-square, etc. $12.00 to $25.00.

Steel tape, 100 feet. $4.00 to $8.00.

An instrument for measuring angles in a horizontal plane. This may be done by means of a plane-table and then measured by a protractor or it may be measured by a graphometer or a surveyor's compass. A more accurate method would be to use a surveyor's transit. The prices of these instruments would be about as follows:

Plane table, $50.00; Graphometer, $25.00; Surveyor's transit (Starret), $16.50.

List for Individual Pupils

The construction work in geometry can be done after a fashion with a ten cent pencil compass and a five cent ruler. But it is hardly fair to the pupil to allow him to do the slovenly, and inaccurate work which alone is possible with such instruments. For the best work each pupil should have:

A pair of compasses with three points, pencil, pen and needle.

A ruling pen.

A bevel edged ruler.

A protractor.

A good scale 6 to 10 inches long.

A triangle with angles 30, 60 and 90 degrees.

Such outfits are put up in neat boxes by the instrument makers at prices varying from $1.00 to $5.00. Even the best of these would not be good enough to serve the purpose of a skilled draughtsman.

The instruments enumerated in any of the above lists may be purchased through school supply dealers of which there are several doing business in this state.